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HOMEWORK 3 ARNAB KAR

1.1.1. $ds^2 = \frac{dx^2 + dy^2}{y^2}$

$$L = \frac{1}{2} \frac{(\dot{x}^2 + \dot{y}^2)}{y^2}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}, \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{y^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = 0 \qquad \frac{y^2 \ddot{x} - 2\dot{x} y \dot{y}}{y^4} = 0$$

$$\ddot{x} - 2\frac{\dot{x}}{y}\dot{y} = 0 \qquad -①$$

$$\frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\partial L}{\partial y}$$

$$\frac{y^2 \ddot{y} - 2y\dot{y}^2}{y^4} = -\frac{(\dot{x}^2 + \dot{y}^2)}{y^3}$$

$$y^2 \ddot{y} - 2y\dot{y}^2 = -\dot{x}^2 y - \dot{y}^2 y$$

$$\ddot{y} - \frac{\dot{y}^2}{y} + \frac{\dot{x}^2}{y} = 0 \qquad -②$$

From ①, ② we find that -

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}, \qquad \Gamma^2_{22} = -\frac{1}{y}, \qquad \Gamma^2_{11} = +\frac{1}{y}.$$

Only derivatives with respect to the (2) component of these is non zero.

$$R^\rho_{22\sigma} = \partial_2 \Gamma^\rho_{2\sigma} - \partial_2 \Gamma^\rho_{2\sigma} + \Gamma^\rho_{2\alpha}\Gamma^\alpha_{2\sigma} - \Gamma^\rho_{2\alpha}\Gamma^\alpha_{2\sigma}$$

But all these combinations turn out to be zero as the terms cancel out.

$$R^\rho_{21\sigma} = \partial_2 \Gamma^\rho_{1\sigma} - \partial_1 \Gamma^\rho_{2\sigma} + \Gamma^\rho_{2\alpha}\Gamma^\alpha_{1\sigma} - \Gamma^\rho_{2\alpha}\Gamma^\alpha_{1\sigma}$$

$\because \Gamma^1_{12}$ is non zero,

$$R^1_{212} = \partial_2 \Gamma^1_{12} - \partial_1 \Gamma^1_{21} + \Gamma^1_{2\alpha}\Gamma^\alpha_{12} - \Gamma^1_{2\alpha}\Gamma^\alpha_{12}$$

$$= \partial_y \left(-\frac{1}{y}\right) - 0 + \Gamma^1_{21}\Gamma^1_{12} - \Gamma^1_{21}\Gamma^1_{12}$$

$$= -\frac{1}{y^2}$$

Since $R^1{}_{212} = -R^1{}_{122} = -\frac{1}{y^2}$.

This is the only non zero component.

$$R^2{}_{121} = \partial_1 \Gamma^2_{21} - \partial_2 \Gamma^2_{11} + \Gamma^2_{1\alpha}\Gamma^\alpha_{21} - \Gamma^2_{2\alpha}\Gamma^\alpha_{12}$$

$$= 0 - \partial_y \left(\frac{1}{y}\right) + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^1_{12}$$

$$= +\frac{1}{y^2}$$

$$R^2{}_{121} = -R^2{}_{122} = \frac{1}{y^2}$$

1.1.2. $$\frac{D^2 v^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma} v^\nu \dot{x}^\rho \dot{x}^\sigma$$

$$\frac{D^2 v^1}{d\tau^2} = -R^1{}_{122} v^1 \dot{y}^2 = -\frac{1}{y^2} v^1 \dot{y}^2$$

$$\frac{d^2 v^2}{d\tau^2} = -R^2{}_{211} v^2 \dot{x}^2 = -\frac{1}{y^2} v^2 \dot{x}^2$$

The sign of the curvature determines whether nearby geodesics will come nearer to each other or go far away.

1.2.1. $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}$$

$$= \partial_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \partial_\nu(\partial_\rho A_\mu - \partial_\mu A_\rho) + \partial_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) A_\rho + (\partial_\nu \partial_\rho - \partial_\rho \partial_\nu) A_\mu + (\partial_\rho \partial_\mu - \partial_\mu \partial_\rho) A_\nu$$

$$= 0$$

We know A's are continuous functions and the order of derivative does not matter. $\partial_\mu \partial_\gamma A_\rho = \partial_\gamma \partial_\mu A_\rho$.

1.2.2. $D_\mu R^\alpha_{\ \gamma\rho\sigma} + D_\gamma R^\alpha_{\ \rho\mu\sigma} + D_\rho R^\alpha_{\ \mu\gamma\sigma} = 0$

$$D_\mu R^\alpha_{\ \gamma\rho\sigma} = \partial_\mu R^\alpha_{\ \gamma\rho\sigma} + \Gamma^\alpha_{\mu\beta} R^\beta_{\ \gamma\rho\sigma} - \Gamma^\beta_{\mu\gamma} R^\alpha_{\ \beta\rho\sigma} - \Gamma^\beta_{\mu\rho} R^\alpha_{\ \gamma\beta\sigma} - \Gamma^\beta_{\mu\sigma} R^\alpha_{\ \gamma\rho\beta}$$

$$D_\gamma R^\alpha_{\ \rho\mu\sigma} = \partial_\gamma R^\alpha_{\ \rho\mu\sigma} + \Gamma^\alpha_{\gamma\beta} R^\beta_{\ \rho\mu\sigma} - \Gamma^\beta_{\gamma\rho} R^\alpha_{\ \beta\mu\sigma} - \Gamma^\beta_{\gamma\mu} R^\alpha_{\ \rho\beta\sigma} - \Gamma^\beta_{\gamma\sigma} R^\alpha_{\ \rho\mu\beta}$$

$$D_\rho R^\alpha_{\ \mu\gamma\sigma} = \partial_\rho R^\alpha_{\ \mu\gamma\sigma} + \Gamma^\alpha_{\rho\beta} R^\beta_{\ \mu\gamma\sigma} - \Gamma^\beta_{\rho\mu} R^\alpha_{\ \beta\gamma\sigma} - \Gamma^\beta_{\rho\gamma} R^\alpha_{\ \mu\beta\sigma} - \Gamma^\beta_{\rho\sigma} R^\alpha_{\ \mu\gamma\beta}$$

Adding all of them -

$$= \partial_\mu R^\alpha_{\ \gamma\rho\sigma} + \partial_\gamma R^\alpha_{\ \rho\mu\sigma} + \partial_\rho R^\alpha_{\ \mu\gamma\sigma} - \Gamma^\beta_{\mu\gamma} R^\alpha_{\ \beta\rho\sigma} - \Gamma^\beta_{\gamma\mu} R^\alpha_{\ \rho\beta\sigma}$$
$$- \Gamma^\beta_{\mu\rho} R^\alpha_{\ \gamma\beta\sigma} - \Gamma^\beta_{\rho\mu} R^\alpha_{\ \beta\gamma\sigma} - \Gamma^\beta_{\gamma\rho} R^\alpha_{\ \beta\mu\sigma} - \Gamma^\beta_{\rho\gamma} R^\alpha_{\ \mu\beta\sigma}$$
$$+ \Gamma^\alpha_{\mu\beta} R^\beta_{\ \gamma\rho\sigma} + \Gamma^\alpha_{\gamma\beta} R^\beta_{\ \rho\mu\sigma} + \Gamma^\alpha_{\rho\beta} R^\beta_{\ \mu\gamma\sigma} - \Gamma^\beta_{\mu\sigma} R^\alpha_{\ \gamma\rho\beta}$$
$$- \Gamma^\beta_{\gamma\sigma} R^\alpha_{\ \rho\mu\beta} - \Gamma^\beta_{\rho\sigma} R^\alpha_{\ \mu\gamma\beta}$$

using $\Gamma^\alpha_{\mu\gamma} = \Gamma^\alpha_{\gamma\mu}$ and $R^\alpha_{\ \rho\beta\sigma} = - R^\alpha_{\ \beta\rho\sigma}$,

$$= \partial_\mu [\partial_\gamma \Gamma^\alpha_{\rho\sigma} - \partial_\rho \Gamma^\alpha_{\gamma\sigma} + \Gamma^\alpha_{\gamma\beta} \Gamma^\beta_{\rho\sigma} - \Gamma^\alpha_{\rho\beta} \Gamma^\beta_{\gamma\sigma}]$$
$$+ \partial_\gamma [\partial_\rho$$

We also consider an inertial frame now where acceleration is zero. Thus in the local intertial frame, ~~due~~ $\Gamma^\alpha_{\mu\nu}$ vanish but not its derivative. Since the above identity is covarient, if it is true in some frame it is true in all frames. Thus we have -

$$= \partial_\mu (\partial_\gamma \Gamma^\alpha_{\rho\sigma} - \partial_\rho \Gamma^\alpha_{\gamma\sigma}) + \partial_\gamma (\partial_\rho \Gamma^\alpha_{\mu\sigma} - \partial_\mu \Gamma^\alpha_{\rho\sigma})$$
$$+ \partial_\rho (\partial_\mu \Gamma^\alpha_{\gamma\sigma} - \partial_\gamma \Gamma^\alpha_{\mu\sigma})$$

$$= (\partial_\mu \partial_\gamma - \partial_\gamma \partial_\mu)\, \Gamma^\alpha_{\rho\sigma} - (\partial_\mu \partial_\rho - \partial_\rho \partial_\mu)\, \Gamma^\alpha_{\gamma\sigma} + (\partial_\gamma \partial_\rho - \partial_\rho \partial_\gamma)\, \Gamma^\alpha_{\mu\sigma}$$

$$= 0$$

The Christoffel symbols are associated with the metric which in turn is continuous.

1.2.3. The trace of the curvature tensor gives Ricci tensor and Ricci scalar. From Bianchi identity –

$$D_\mu R^\alpha{}_{\gamma\rho\sigma} + D_\gamma R^\alpha{}_{\rho\mu\sigma} + D_\rho R^\alpha{}_{\mu\gamma\sigma} = 0$$

Multiplying by $g_{\delta\alpha}$, we note that –

$$D_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^\beta_{\mu\rho} g_{\beta\sigma} - \Gamma^\beta_{\mu\sigma} g_{\rho\beta} = 0$$

(covariant derivative of metric is zero)

$$D_\mu R_{\gamma\rho\sigma\delta} + D_\gamma R_{\rho\mu\sigma\delta} + D_\rho R_{\mu\gamma\sigma\delta} = 0$$

Multiplying by $g^{\mu\sigma}$ –

$$D^\sigma R_{\gamma\rho\sigma\delta} + D_\gamma R_{\rho\delta} - D_\rho R_{\gamma\delta} = 0$$

Multiplying by $g^{\gamma\delta}$ –

$$D^\sigma R_{\rho\sigma} + D^\delta R_{\rho\delta} - D_\rho R^\delta_\delta = 0$$

$$D^\sigma R_{\rho\sigma} + D^\sigma R_{\rho\sigma} = D_\rho R$$

$$D^\sigma R_{\rho\sigma} = \frac{1}{2} D_\rho R = \frac{1}{2} D^\sigma g_{\sigma\rho} R$$

$$D^\sigma \left(R_{\rho\sigma} - \frac{1}{2} g_{\sigma\rho} R \right) = 0$$

$$D^\mu \left(R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R \right) = 0$$

$$G_{\mu\gamma} = R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R = R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} g^{\alpha\beta} R_{\alpha\beta}$$

$G_{\mu\gamma}$ is the required tensor.

1.3.2. $S = \frac{1}{4} \int F_{\mu\gamma} \eta^{\mu\alpha} \eta^{\gamma\beta} F_{\alpha\beta} \, dx$

$= \frac{1}{4} \int \eta^{\mu\alpha} \eta^{\gamma\beta} F_{\mu\gamma} F_{\alpha\beta} \, dx$

$\delta S = \frac{1}{4} \int \eta^{\mu\alpha} \eta^{\gamma\beta} \delta F_{\mu\gamma} F_{\alpha\beta} \, dx + \frac{1}{4} \int \eta^{\mu\alpha} \eta^{\gamma\beta} F_{\mu\gamma} \delta F_{\alpha\beta} \, dx$

$\because \eta^{\mu\alpha} = \eta^{\alpha\mu}$ interchanging μ, α and γ, β in last term -

$= \frac{1}{2} \int \eta^{\mu\alpha} \eta^{\gamma\beta} \delta F_{\mu\gamma} F_{\alpha\beta} \, dx$

$\delta F_{\mu\gamma} = \partial_\mu \delta A_\gamma - \partial_\gamma \delta A_\mu + \delta A_\mu A_\gamma + A_\mu \delta A_\gamma - \delta A_\gamma A_\mu - A_\gamma \delta A_\mu$

$= \partial_\mu (\partial_\rho A_\gamma \delta x^\rho) - \partial_\gamma (\partial_\rho A_\mu \delta x^\rho) + (\partial_\rho A_\mu) A_\gamma \delta x^\rho + A_\mu (\partial_\rho A_\gamma) \delta x^\rho - (\partial_\rho A_\gamma) A_\mu \delta x^\rho - A_\gamma (\partial_\rho A_\mu) \delta x^\rho$

$= (\partial_\mu \partial_\rho A_\gamma) \delta x^\rho + (\partial_\rho A_\gamma)(\partial_\mu \delta x^\rho) - (\partial_\gamma \partial_\rho A_\mu) \delta x^\rho - (\partial_\rho A_\mu)(\partial_\gamma \delta x^\rho) + [(\partial_\rho A_\mu) A_\gamma + A_\mu \partial_\rho A_\gamma - (\partial_\rho A_\gamma) A_\mu - A_\gamma (\partial_\rho A_\mu)] \delta x^\rho$

Let us consider integral of this type -

$\int f(x) \, \partial_\gamma \delta x^\rho \, dx$

integration computed with respect to γ component -

$= f(x) \, \delta x^\rho \Big| - \int \partial_\gamma f(x) \, \delta x^\rho \, dx^\gamma$

Using the above facts we have -

$\delta S = \frac{1}{2} \int \eta^{\mu\alpha} \eta^{\gamma\beta} F_{\alpha\beta} \big((\partial_\mu \partial_\rho A_\gamma - \partial_\gamma \partial_\rho A_\mu + (\partial_\rho A_\mu) A_\gamma + A_\mu \partial_\rho A_\gamma$

$- (\partial_\rho A_\gamma) A_\mu - A_\gamma (\partial_\rho A_\mu)) \delta x^\rho \big) \, dx$

$+ \frac{1}{2} \int \eta^{\mu\alpha} \eta^{\gamma\beta} F_{\alpha\beta} (\partial_\mu \partial_\rho A_\gamma - \partial_\gamma \partial_\rho A_\mu) \delta x^\rho dx$

Since this result $\delta S = 0$ for any δx^ρ, integrand is zero.

$$\eta^{\mu\alpha}\eta^{\gamma\beta}F_{\alpha\beta}\big(\partial_\mu\partial_\rho A_\gamma + (\partial_\rho A_\mu)A_\gamma + A_\mu(\partial_\rho A_\gamma)$$
$$- (\partial_\rho A_\gamma)A_\mu - A_\gamma(\partial_\rho A_\mu) - \partial_\gamma\partial_\rho A_\mu + \partial_\mu\partial_\rho A_\gamma - \partial_\gamma\partial_\rho A_\mu\big) = 0$$

$$F^{\mu\gamma}\big(2\partial_\mu\partial_\rho A_\gamma - 2\partial_\gamma\partial_\rho A_\mu + (\partial_\rho A_\mu)A_\gamma - (\partial_\rho A_\gamma)A_\mu$$
$$+ A_\mu\partial_\rho A_\gamma - A_\gamma\partial_\rho A_\mu\big) = 0$$

$$F^{\mu\gamma}(\partial_\rho A_\gamma)A_\mu$$
$$= F^{\gamma\mu}(\partial_\rho A_\mu)A_\gamma = -F^{\mu\gamma}(\partial_\rho A_\mu)A_\gamma$$

Thus we get -

$$F^{\mu\gamma}\big(\partial_\mu\partial_\rho A_\gamma - \partial_\gamma\partial_\rho A_\mu + (\partial_\rho A_\mu)A_\gamma + A_\mu\partial_\rho A_\gamma\big) = 0$$

$$F^{\mu\gamma}\big(2\partial_\mu\partial_\rho A_\gamma + \partial_\rho(A_\mu A_\gamma)\big) = 0$$

$$F^{\mu\gamma}\big(\partial_\mu\partial_\rho A_\gamma + \tfrac{1}{2}\partial_\rho(A_\mu A_\gamma)\big) = 0$$

1.3.1. $F_{\mu\gamma} = \partial_\mu A_\gamma - \partial_\gamma A_\mu + [A_\mu, A_\gamma]$

Let us define covarient derivative -

$$D_\rho = \partial_\rho + A_\rho$$

$$[D_\mu, D_\gamma] = [\partial_\mu + A_\mu, \partial_\gamma + A_\gamma]$$
$$= \partial_\mu\partial_\gamma - \partial_\gamma\partial_\mu + [A_\mu, A_\gamma] + [A_\mu, \partial_\gamma]$$
$$+ [\partial_\mu, A_\gamma]$$

$$[A_\mu, \partial_\gamma]\psi$$
$$= A_\mu\partial_\gamma\psi - A_\mu(\partial_\gamma\psi) - (\partial_\gamma A_\mu)\psi = -\partial_\gamma A_\mu\psi$$

$$[D_\mu, D_\gamma] = +[A_\mu, A_\gamma] + (-\partial_\gamma A_\mu) + (\partial_\mu A_\gamma)$$
$$= +(\partial_\mu A_\gamma - \partial_\gamma A_\mu + [A_\mu, A_\gamma])$$
$$= F_{\mu\gamma}$$

The analouge of Bianchi identity is -

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$$

To arrive at this we use Jacobi identity –

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0$$

$$[D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0$$

we now see its action on a wave function –

$$[D_\mu, F_{\nu\rho}]\psi + [D_\nu, F_{\rho\mu}]\psi + [D_\rho, F_{\mu\nu}]\psi = 0$$

$$D_\mu(F_{\nu\rho}\psi) - F_{\nu\rho} D_\mu\psi + D_\nu(F_{\rho\mu}\psi) - F_{\rho\mu} D_\nu\psi + D_\rho(F_{\mu\nu}\psi) - F_{\mu\nu} D_\rho\psi = 0$$

$$(D_\mu F_{\nu\rho})\psi + F_{\nu\rho} D_\mu\psi - F_{\nu\rho} D_\mu\psi + (D_\nu F_{\rho\mu})\psi + F_{\rho\mu} D_\nu\psi - F_{\rho\mu} D_\nu\psi + (D_\rho F_{\mu\nu})\psi + F_{\mu\nu} D_\rho\psi - F_{\mu\nu} D_\rho\psi = 0$$

$$(D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu})\psi = 0$$

$\because$ This is true for any ψ, we get –

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$$