PHY510 Relativistic Quantum Mechanics 2004 S. G. Rajeev

Problem Set 1 Due Feb 15 2007

1 Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ be the usual Pauli matrices and let $B = \cos\theta\sigma_1 + \sin\theta\sigma_2$.

1.1 Suppose we measure the observable represented by σ_1 and the outcome is 1. The instant after this, B is measured. What are the possible outcomes? What is the probability of each outcome?

1.2 Suppose the hamiltonian of a system is given by $H = \sigma_1 B_1 + \sigma_2 B_2 + \sigma_3 B_3$ for some vector B. At time $t = 0$ a measurement of σ_3 yields the value 1. What is the probability of each outcome if σ_3 is measured again at a later time t?

2 Recall that every rotation matrix $R \in SO(3)$ corresponds to two matrices in $SU(2)$, given by the relation $g\sigma \cdot vg^\dagger = \sigma \cdot (Rv)$.

2.1 When R is a rotation through an angle θ around the third axis, find the solutions for g of the above equation. What happens as θ varies from 0 to 2π?

2.2 A spinor $\psi : R^3 \to C^2$ transforms according to $\psi(x) \mapsto g\psi(R^{-1}x)$ under a rotation.Show that the norm of the spinor $|\psi|^2$ is unchanged under this transformation.

2.3 By considering infinitesimal rotations, show that a particle whose wavefunction is represented by such a spinor has an intrinsic angular momentum $\frac{\hbar}{2}$ when it is at rest.

2.4 Let P be the matrix representing parity $Px = -x$.Can you represent the effect of parity by a 2×2 matrix acting on spinors?

3 Recall that energy and momentum together transform as a four-vector under Lorentz transformations; also, energy and momentum are both conserved in particle decays and collisions. (Contd. in next page)

3.1 Show that a massive particle cannot decay into a single photon (which is massless).

3.2 When a massive particle of mass m, at rest, decays into two photons, what is the magnitude of the momentum of each of the photons?

3.3 An electron of momentum p and mass m collides with a nucleus of mass $M > m$ at rest. After the collision electron leaves at an angle θ relative to the incident direction. Find the momentum of the electron after the collision, as well as the momentum and direction of motion of the recoiling nucleus.

PHY510 Relativistic Quantum Mechanics 2004 S. G. Rajeev

Problem Set 2 Due Mar 8 2007

4 Define the Dirac matrices by $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$ as usual.

4.1 Show that the matrices $\frac{1}{2}\sigma_{\mu\nu}$, where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, satisfy the commutation relations of the Lorentz Lie algebra:

$$[\frac{1}{2}\sigma_{\mu\nu}, \frac{1}{2}\sigma_{\rho\sigma}] = \eta_{\nu\rho}\frac{1}{2}\sigma_{\mu\sigma} - \eta_{\mu\rho}\frac{1}{2}\sigma_{\nu\sigma} - \eta_{\nu\sigma}\frac{1}{2}\sigma_{\mu\rho} + \eta_{\mu\sigma}\frac{1}{2}\sigma_{\nu\rho} \tag{1}$$

4.2 Prove that the Dirac matrices transform as the components of a four-vector under this representation:

$$[\frac{1}{2}\sigma_{\mu\nu}, \gamma_\sigma] = \eta_{\nu\sigma}\gamma_\mu - \eta_{\mu\sigma}\gamma_\nu \tag{2}$$

4.3 Show that

$$J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + \frac{1}{2}\sigma_{\mu\nu} \tag{3}$$

also form a representation of the Lorentz algebra under which the Dirac operator is invariant:

$$[J_{\mu\nu}, \gamma\cdot\partial] = 0. \tag{4}$$

5 Find a matrix P such that

$$P\gamma_0 P^{-1} = \gamma_0, \quad P\gamma_1 P^{-1} = -\gamma_1, \quad P\gamma_2 P^{-1} = -\gamma_2, \quad P\gamma_3 P^{-1} = -\gamma_3 \tag{5}$$

and another satisfying

$$C\gamma_\mu C^{-1} = -\gamma_\mu^\dagger. \tag{6}$$

Show that these lead to parity and charge conjugation invariance for the Dirac equation.

PHY510 Relativistic Quantum Mechanics 2004 S. G. Rajeev

Problem Set 3 Due Apr 3rd 2007

6 Reduce the radial part of the Dirac equation in a Coulomb potential to the confluent hypergeometric equation by an appropriate change of variables. **Hint**: You may use notes from the class, but give details of the derivation.

6.1 Determine the eigenvalues of energy corresponding to the solutions that vanish at infinity (the 'bound states'). Compare with the known spectrum of hydrogen.

7 Determine the behavior at infinity of the series

$$F_1^1(a,c;z) = 1 + \frac{a}{c}z + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \tag{7}$$

What is special about the case where a equals a negative integer?

8 Derive an integral representation of the form $F_1^1(a,c;z) = \int_C e^{tz}v(t)dt$ for some function $v(t)$ and a suitable contour on the t-plane.**Hint**: derive a first order differential equation for $v(t)$, solve it and then study which contour will make the integral converge. This is called Laplace's method.

PHY510 Relativistic Quantum Mechanics 2004 S. G. Rajeev

Problem Set 4 Due Apr 17th 2007

9 Consider matter whose equation of state is

$$P = K\nu^{1+\frac{1}{n}} \tag{8}$$

where P is the pressure and ν is the number density. The constant n is called the *polytropic index*. Determine this index n and the constant K for (i) a degenerate non-relativistic free fermi gas of electrons of mass m and (ii) a degenerate relativistic electron gas of negiligible mass.

10 Assume that the electrons have an equation of state as above with index n.

10.1 Show that the condition for equilibrium of a white dwarf is

$$-\nabla P = \rho(x)\nabla\phi \tag{9}$$

where ϕ is the gravitational potential, P is the pressure of the electron gas, and $\rho(x) = m_p\nu(x)$ is the mass density due to protons.

10.2 Using Poisson's equation $\nabla^2\phi = 4\pi G\rho$, and the equation of state of the electron gas, derive a non-linear partial differential equation for the gravitational potential ϕ.

10.3 Get an ordinary differential equation by assuming that the potential depends only on the distance from the center of the star; (e.g., this ignores rotation). What are the boundary conditions at the center and at the surface of the star?

11 The Lane-Emden equation of index n is defined by

$$\theta''(\xi) + \frac{2}{\xi}\theta' + \theta^n(\xi) = 0, \quad \theta(0) = 1, \quad \theta'(0) = 0. \tag{10}$$

11.1 Find a change of variables from $\phi(r)$ to $\theta(\xi)$ that turns the equation above for a white dwarf into the Lane-Emden equation . Take into account also the required boundary conditions at the center of the star and its surface.

continued next page..

12 Solve the Lane-Emden equation **numerically** (for the appropriate value of n for a relativistic electron gas) to find the smallest value of ξ at which $\theta(\xi) = 0$. Use this to find the mass of the white dwarf in this approximation. (A more detailed analysis shows that this is in fact an upper limit, but I am not asking you to show that.)

PHY510 Relativistic Quantum Mechanics 2004 S. G. Rajeev

Problem Set 5 Due May 1st 2007

13 Let

$$H = \omega b^\dagger b + \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_1^\dagger a_2 + g b a_2^\dagger a_1 + g^* b^\dagger a_1^\dagger a_2 \tag{11}$$

be the hamiltonian of a two level atom interacting with a photon.

13.1 Find the exact answer for the energy difference between the ground state and the first excited state as a function of g.

13.2 Compare this to the answer you would have obtained in the second order of perturbation theory in g.

14 A spin zero particle of mass m interacts with a fermion that can exist in one of two possible states. Assume that both states of the fermion are much heavier than the boson. Transition between the fermionic states is possible by emission or absorption of a virtual boson. Construct a hamiltonian that describes this system. Find its ground state and first excited state in a suitable approximation.

I have kept this problem vague to give you a chance to formulate and solve a simple theoretical model on your own. Not everyone needs to find exactly the same model. Look at T. D. Lee Phys. Rev. **45** 1329 (1954). The more adventurous should find variants of the Lee model that are still solvable.