Syllabus

•The Schrödinger operator; heat kernel; harmonic oscillator.

•Intuitive formulation of the Feynman-Kac formula; Wiener integrals; path integrals.

•Gaussian Integrals; Moments; Determinants; Wick's theorem; perturbation theory.

•DeWitt's definition of the Determinants of operators; path integral solution of the harmonic oscillator.

•Feynman's formulation of quantum mechanics.

•Grassmann variables; fermionic path integrals.

•Classical fields; Klein-Gordon equation; Maxwell's equations; Dirac equation.

•Free Quantum fields; Green's functions; Feynman's interpretation of Green's functions.

•Advanced Topic: Schwinger's solution of Quantum Electrodynamics in constant external background.

•Interacting fields; perturbation theory; Feynman diagrams.

•Anamolous magnetic moment; Lamb shift.

•Critical phenomena.

Chapter 1

Introduction

1 Field theory is the mathematical apparatus of the physics of elementary particles. It originated in the quantum theory of light, *Quantum Electrodynamcs*.

2 It describes a quantum system whose dynamical variables depend on time and space co-ordinates. Such variables are called *fields*. The predictions of the theory are the expectation values of products of these field variables.

2.1 Field theory is to particle physics what calculus is to mechanics. Currently the subject is in a primitive stage, essentially just a set of rules of calculation (which give amazingly accurate results). It is a major challenge the 21st century to put this theory on a sound mathematical foundation just as calculus was incorporated into modern analysis in the nineteenth cnetury.

2.2 Wilson found that field theory is also the correct mathematical description of second order phase transitions. Witten has shown that field theory is a powerful mathematical tool in pure mathematical areas as well, such as knot theory, algebraic geometry and the classification of four-manifolds. Ideas that originated in field theory have found applications to diverse areas: renormalization in dynamical systems (Feigenbaum, McMullen); vertex operator algebras in Lie theory(Lepowsky, Borcherds); deformation quantization in symplectic geomery (Kontsevich); quantum groups in integrable systems(Faddeev, Drinfeld)...

2.3 Quantum Electrodynamics, the original example of a field theory, was invented by Dirac, Tomonaga, Feynman and Schwinger in the nineteen forties. It describes the interaction of electromagnetic radiation (light) with electrons and atoms with an amazing accuracy:the level of 12 decimal places in certain parameters. It is the most accurate physical theory ever.

3 The path integral method of Feynman is the most elegant existing method to understand field theory. The idea is to incorporate quantum fluctuations by averaging over all possible values of the field variables. This requires some notion of integration on an infinite dimensional space.

3.1 We will develop it first in the more familiar context of quantum mechanics. Quantum mechanics can be thought of as field theory in one dimension: all the dynamical variables depend on just one variable,time.

Chapter 2

The Schrödinger Operator

4 The classic paper *R. P. Feynman, Space-time Approach to Non-Relativistic Quantum Mechanics, Rev. Mod. Phys. Vol. 20, p. 267 (1948)* is the source of this material. To this day there is no substitute for the physical insights of this paper..

5 We begin with the most familiar formulation of quantum mechanics, in terms of the *Schrödinger equation*:

$$[-\nabla^2 + V]\psi = i\frac{\partial \psi}{\partial t}.$$

5.1 Here, the *wave function* $\psi : Q \times R \to C$ depends on the position variables of the system $q \in Q$ as well as time $t \in R$. The potential energy $V : Q \to R$ is a function of the position alone. ∇^2 is the Laplace operator on the configuration space.

5.2 To define the Laplace operator we will need a Riemannian metric on Q. For simplicity we will assume that $Q = R^n$ with the Euclidean metric.

6 The operator $H = -\nabla^2 + V$ is called the *Hamiltonian* or the *Schrödinger operator*. If V is not too singular, H is a self-adjoint operator on the complex Hilbert space $L^2(Q)$ of wavefunctions. This it is an observable, with the physical meaning of energy. We will assume in addition that its spectrum is bounded below, so that there is a state (the *ground state*) with the lowest value for energy.

7 Given the initial condition for the wavefunction $\psi(0)$ at $t = 0$, we can solve the Schrödinger equation to get the wave function at later times:

$$\psi(t) = e^{-iHt}\psi(0).$$

The unitary operator $K_t = e^{-iHt}$ is called the *propagator* as it takes the system from the initial time 0 to time t.

7.1 In more detail,

$$\psi(t,q) = \int K_t(q,q')\psi(t,q')dq'.$$

Here, the integral kernel of K_t is

$$K_t(q,q') =< q|e^{-iHt}|q' > .$$

It satisfies the differential equation

$$[-\nabla_q^2 + V(q)]K_t(q,q') = i\frac{\partial}{\partial t}K_t(q,q')$$

with the initial condition

$$\lim_{t\to 0} K_t(q,q') = \delta(q-q').$$

8 The unitary operator $K_t = e^{-iHt}$ can be thought of as analytic continuation of the positive operator $h_t = e^{-Ht}$ (defined for $t \geq 0$) to the imaginary axis. Although K_t is the one with the direct physical significance, it is mathematically much more convenient to work with h_t.

8.1 For small values of t $K_t(q,q')$ is a rapidly oscillating function of t which averages out to the delta function. But h_t is much more well-behaved, as the following example will show.

9 For the case of the free particle ($V = 0$), we can solve the Schrödinger equation by decomposing into plane waves (Fourier analysis) [1]:

$$K_t(q,q') = \int [dk]e^{ik\cdot(q-q')}e^{-ik^2t}, \quad h_t(q,q') = \int [dk]e^{ik\cdot(q-q')}e^{-k^2t},$$

[1] $[dk]$ denotes $\frac{d^nk}{[2\pi]^n}$ where n is the dimension of the configuration space.

so that

$$K_t(q,q') = \frac{1}{[4\pi i t]^{\frac{n}{2}}} e^{i\frac{(q-q')^2}{4t}}, \quad h_t(q,q') = \frac{1}{[4\pi t]^{\frac{n}{2}}} e^{-\frac{(q-q')^2}{4t}}.$$

10 The *reproducing property* of the kernel follows from $e^{-H(t-t_1)}e^{-Ht_1} = e^{-Ht}$:

$$h_t(q,q_0) = \int h_{t-t_1}(q,q_1) h_{t_1}(q_1,q_0) dq_1$$

when $t > t_1 > 0$. By iteration, we get, for $t > t_n > t_{n-1} \cdots t_1 > 0$,

$$h_t(q,q_0) = \int h_{t-t_n}(q,q_n) h_{t_n-t_{n-1}}(q_n,q_{n-1}) \cdots h_{t_1} q_1, q_0) dq_1 dq_2 \cdots dq_n.$$

10.1 We are subdividing the interval $[0,t]$ into smaller intervals. For example the subintervals could be of equal length; by choosing n big enough we can write h_t in terms of the heat kernel for arbitrarily small intervals.

11 When t is very small, we can approximate

$$e^{-[-\nabla^2+V]t} = e^{-\frac{1}{2}Vt} e^{\nabla^2 t} e^{-\frac{1}{2}Vt}[1 + O(t^2)].$$

In other words

$$h_t(q,q') \sim \frac{1}{[4\pi t]^{\frac{n}{2}}} e^{-\frac{(q-q')^2}{4t}} e^{-\frac{1}{2}[V(q)+V(q')]t}[1 + \mathrm{O}(t^2)]$$

12 Another way to state this result is in terms of the Trotter product formula

$$e^{A+B} = \lim_{N\to\infty} e^{\frac{B}{2N}} e^{\frac{A}{N}} e^{\frac{B}{2N}}.$$

This can be proved by operator methods (see e.g., Glimm and Jaffe, *Quantum Theory: a Path Integral Point of View*).

12.1 This can also be proved by substituing the ansatz

$$h_t(q,q') = \frac{e^{\frac{-(q-q')^2}{4t}}}{[4\pi t]^{\frac{n}{2}}}\left[\sum_{r=0}^{\infty} a_r(q,q')t^n\right]$$

into the 'heat equation'

$$[-\nabla^2 + V]h_t = -\frac{\partial h_t}{\partial t}.$$

This gives a series of recursion relations for a_r ; $a_0(q,q') = 1, a_1(q,q') = -\frac{1}{2}[V(q)+V(q')]$ etc. follow.

13 Now we can combine these results to get

$$h_t(q,q_0) = \lim_{\epsilon\to 0, N\to\infty, N\epsilon=t} \int e^{-\left[\frac{1}{4\epsilon}\sum_{k=1}^{N}(q_k-q_{k-1})^2+\epsilon\sum_{k=1}^{N}V(q_k)+\frac{1}{2}\epsilon[V(q)+V(q_0)]\right]} \frac{d^n q_1}{[4\pi\epsilon]^{\frac{n}{2}}} \cdots \frac{d^n q_N}{[4\pi\epsilon]^{\frac{n}{2}}}$$

14 Note that

$$\lim_{\epsilon\to 0, N\to\infty, N\epsilon=t} \epsilon\sum_{k=1}^{N} V(q_k) = \int_0^t V(q(\tau))d\tau,$$

$$\lim_{\epsilon\to 0, N\to\infty, N\epsilon=t} \epsilon\sum_{k=1}^{N} \frac{(q_k-q_{k-1})^2}{\epsilon} = \int_0^t \dot{q}(\tau)^2 d\tau$$

since q_k is the position of the particle at time t_k . Thus in the limit we can imagine that we are integrating over an infinite dimensional space, the space of all continuous curves that start at q_0 at time 0 and end at q at time t . We write

$$h_t(q,q_0) = \int_{\substack{q(0)=q_0\\ q(t)=q}} e^{-\int_0^t\left[\frac{1}{4}\dot{q}^2(\tau)+V(q(\tau))\right]d\tau}\mathcal{D}q.$$

14.1 We use the notation $\mathcal{D}q$ to denote the limit of the integration measures $\frac{d^n q_1}{[4\pi\epsilon]^{\frac{n}{2}}} \cdots \frac{d^n q_N}{[4\pi\epsilon]^{\frac{n}{2}}}$. The idea here is similar to the way we define an integral as the limit of a sum multiplied by an infinitesimally small constant. Later we will describe how to make precise mathematical sense of such a measure of integration on such an infinite dimensional space.

15 To get a physical interpretation, let us transform to the case of the Schrd̈inger kernel, ($t \to it$) and also put back the constants [2]

$$K_t(q, q_0) = \int_{\substack{q(0) = q_0 \\ q(t) = q}} e^{\frac{i}{\hbar}\left\{ \int_0^t \left[\frac{1}{2} m\dot{q}^2(\tau) - V(q(\tau))\right] d\tau \right\}} \mathcal{D}q.$$

15.1 The quantity in the curly brackets is just the action.

16 Thus the propagator is the average over all paths, with a weight for each path equal to $e^{\frac{i}{\hbar}S[q]}$, $S[q]$ being the action of the path. This is known as *Feynman's action principle*.

17 In the limit as $\hbar \to 0$, the integrand oscillates rapidly; the largest contribution will come from the neighborhood of the path for which the phase is stationary: $\delta S[q] = 0$. This is the classical path. Thus we recover classical mechanics in the limit as $\hbar \to 0$.

17.1 We will return to this steepest descent approximation later, working it out in much more detail.

[2] We had chosen units such that $\hbar = 1, 2m = 1$.

Chapter 3

Random Variables and Their Moments

18 A good reference for this material is *L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge, 1995, Chapter 1.* There are many other standard textbooks on random variables as well.

19 We saw that quantum mechanics leads naturally to questions about integrals on infinite dimensional spaces. In imaginary time, these integrals have a positive density; they can be viewed as probability distributions.

20 We will first study a finite dimensional model, integrals of the form

$$\int e^{-S(\phi)} f(\phi) d^n\phi$$

where $S(\phi)$ is a real function $S : R^n \to R$ such that the integral $\int e^{-S(\phi)} d^n\phi$ converges. Also, $f : R^n \to R$ is a function, for example a polynomial.

21 The quantity $S(\phi)$ will be called the *action* (by analogy with the language we will eventually develop for field theory). The *expectation value* of the function f is

$$< f >= \frac{\int e^{-S(\phi)} f(\phi) d^n\phi}{\int e^{-S(\phi)} d^n\phi}.$$

21.1 Clearly, $\frac{e^{-S(\phi)}d^n\phi}{\int e^{-S(\phi)}d^n\phi}$ is a probability measure on R^n . Thus ϕ and $f(\phi)$ are random variables, whose the expectation value is as defined above.

22 The *moments* of the random variable ϕ are the symmetric tensors

$$G^{i_1\cdots i_n} =< \phi^{i_1} \cdots \phi^{i_n} > .$$

The *characteristic function* or *moment generating function* is the Fourier transform

$$Z(J) = \int e^{-S(\phi)+iJ_k\phi^k} d^n\phi.$$

By expanding, we see that

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{1}{n!} G^{k_1\cdots k_n} iJ_{k_1} \cdots iJ_{k_n}.$$

22.1 Of special interest are the *mean* $< f >$ and *variance*

$$< [f- < f >]^2 >=< f^2 > - < f >^2 .$$

of a random variable f . The mean measures its typical value and the variance the deviation from the typical value.

23 The simplest example is when the action is a quadratic function: $S(\phi) = \frac{1}{2}A_{ij}\phi^i\phi^j$, with A being a positive matrix. This is the case of a Gaussian integral. More complicated cases, are usually thought of as perturbations around this one.

23.1 We will see that Gaussian integrals over infinite dimensional spaces descibe simple systems such as the free particle, the harmonic oscillator, or any system of free bosons. The finite dimensional integrals we study here are useful as toy models.

23.2 The simplest case is the integral with just two variables (even simpler than the case with one variable!):

$$\int e^{-\frac{1}{2}(x^2+y^2)} dxdy = 2\pi \int_0^\infty e^{-\frac{1}{2}r^2} rdr = 2\pi.$$

We conclude from this that

$$\int e^{-\frac{1}{2}x^2}dx = \sqrt{[2\pi]}, \int e^{-\frac{1}{2}ax^2}dx = \sqrt{[\frac{2\pi}{a}]}, a > 0.$$

and more generally that

$$\int e^{-\frac{1}{2}\sum_{i=1}^{n}[a_i\phi^i]^2}d^n\phi = [2\pi]^{\frac{n}{2}}[\prod_i^n a_i]^{-\frac{1}{2}}.$$

23.3 Any positive matrix can be diagonalized by special orthogonal transformations $A = SDS^T$, where the diagonal matrix is given by the eigenvalues of A : $D = \mathrm{diag}\{a_1, \cdots a_n\}$. Moreover $SS^T = 1$ which implies that $|\det S| = 1$. Thus by a linear transformation $\chi = S\phi$ the more general Gaussian integral can be evaluated:

$$\int e^{-\frac{1}{2}\phi^T A\phi}d^n\phi = \int e^{-\frac{1}{2}\sum_i a_i\chi_i^2}d^n\chi = [2\pi]^{\frac{n}{2}}[\prod_i^n a_i]^{-\frac{1}{2}} = \sqrt{\det[2\pi A^{-1}]}$$

23.4 Now let us look at the moments of a Gaussian. Starting with the case of a single variable, it is elementary that

$$\begin{aligned}
\int e^{-\frac{1}{2}x^2}x^{2n-1}dx &= 0, \\
\int e^{-\frac{1}{2}x^2}x^{2n}dx &= 2\int_0^\infty e^{-u}[2u]^{n-\frac{1}{2}}du \\
&= 2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2}) \\
&= \sqrt{2}\ 2^n(n-\frac{1}{2})(n-\frac{3}{2})\cdots\Gamma(\frac{1}{2}) \\
&= \sqrt{[2\pi]}(2n-1)!!.
\end{aligned}$$

The moment $< x^{2n} >= (2n-1)!!$ is the number of partitioning a set of $2n$ elements into blocks of two elements each. The reason for this combinatorial meaning will be clear soon.

23.5 Another way is to evaluate the Fourier transform by 'completion of squares', $y = x - iJ$

$$\int e^{-\frac{1}{2}x^2+ixJ}dx = e^{-\frac{1}{2}J^2}\int e^{-\frac{1}{2}y^2}dy = \sqrt{[2\pi]}e^{-\frac{1}{2}J^2}.$$

The above moments can be obtained by expanding in powers of J .

24 By the transformation $\chi = \phi - iA^{-1}J$, we get

$$\int e^{-\frac{1}{2}\phi^T A\phi + i\phi^T J} d^n\phi = e^{-\frac{1}{2}J^T A^{-1} J}\sqrt{\det[2\pi A^{-1}]}.$$

25 By expanding $Z(J)$ to second order we get

$$< \phi^i >= 0, \quad < \phi^i \phi^j >= C^{ij}$$

where C (called the *covariance matrix*) is the inverse of A .

26 *Wick's Theorem:* The moments of a Gaussian random variable with zero mean and covariance C are given by

$$< \phi^{i_1} \cdots \phi^{i_n} >= \sum_{\pi \in P_2(n)} \prod_{\{a,b\}\in\pi} C^{i_a i_b}.$$

Here, π is a partition of the set $\{1, 2, 3, \cdots n\}$ into subsets of two elements each; and $P_2(n)$ is the set of all such partitions.

26.1 This theorem is also known as the *Gaussian moment theorem* in other parts of physics, e.g., in Quantum Optics. See the classic book by *Mandel and Wolf, Ch. 1*

26.2 If n is odd it is not possible to partition into pairs, so the moment is zero.

26.3 For example the partitions into pairs of a set of four elements are

$$P_2(4) = \{\ \{\{1,2\},\{3,4\}\},\ \ \{\{1,3\},\{2,4\}\},\ \ \{\{1,4\},\{2,3\}\},\ \}$$

which gives

$$< \phi^{i_1}\phi^{i_2}\phi^{i_3}\phi^{i_4} >= C^{i_1 i_2}C^{i_3 i_4} + C^{i_1 i_3}C^{i_2 i_4} + C^{i_1 i_4}C^{i_2 i_4}.$$

The next case (the sixth moment) is a sum of 15 terms, the eighth moment a sum of 105 terms and so on.

26.4 To prove this theorem, we note that the moments transform like the components of a completely symmetric tensor of order n . So we can choose a basis that is convenient for our purposes: the one in which the covariance matrix is diagonal. (In fact, since $C > 0$, we can even choose it to be the identity using transformations that are not orthogonal .) Then the variables ϕ^i become independent random variables: the probability distribution is a product $\prod_i e^{-\frac{1}{2}\phi_i^2}$. Thus the moments are products of the moments for each variable separately: the problem reduces to the case of a single variable. But the theorem is obvious in the case of a single variable, since $(2n-1)!!$ is the number of pairings.

26.5 A more explicit proof is to differentiate the generating function with respect to J and then setting J equal to zero.

26.6 Yet another proof is to introduce a set of operators satisfying the *Canonical Commutation Relations*:

$$[a_i, a^{\dagger j}] = \delta_i^j, \quad [a_i, a_j] = [a^{\dagger i}, a^{\dagger j}] = [a_i, a_j] = 0$$

An explicit realization of these relations is

$$a_i = 2\frac{\partial}{\partial \phi^i} + \frac{1}{2}A_{ik}\phi^k, \quad a^{\dagger i} = -2C^{ik}\frac{\partial}{\partial \phi^k} + \frac{1}{2}\phi^i.$$

There is a state satisfying $a_i\psi_0 = 0$, the *vacuum state* $\psi_0(\phi) = e^{-\frac{1}{4}\phi^T A\phi}$. The expectation value with respect to the Gaussian measure is just the vacuum expectation value in this sense:

$$< \phi^{i_1} \cdots \phi^{i_n} >= \frac{< \psi_0, [a^{\dagger i_1} + C^{i_1 j_1}a_{j_1}] \cdots [a^{\dagger i_n} + C^{i_n j_n}a_{j_n}]\psi_0 >}{< \psi_0, \psi_0 >}$$

26.7 We say that any monomial (of non-zero order)in $a, a^\dagger$ is *normal ordered* or *Wick ordered* if the *creation operators* $a^\dagger$ stands to the left of the *annihilation operators* a . By repeated use of the commutation relations, we can rewrite any polynomial in $a, a^\dagger$ as a sum of normal ordered monomials. Moreover the vacuum expectation value of any normal ordered monomial is zero.

26.8 Returning to the calculation of the moments, we can imagine moving the annihilation operator to the right until they reach the vacuum state to give zero. Each time they pass through a creation operator we get a factor C . This way we see that the vacuum expectation value is a sum over products of C : one factor of C for each pair and one such term for each partition into pairs. See *Glimm and Jaffe, Quantum Physics* for a more detailed presentation of these ideas.

27 Yet another way to understand Wick's theorem is in terms of symmetries. It is clear that the answer can only depend on A_{ij} . The group that leaves it invariant is the orthogonal group. The only invariant symmetric tensor of the orthogonal group is the one that appears in Wick's theorem; the overall constant has to be determined by computing the special case $<[\phi^2]^n>$, where $\phi^2 = A_{ij}\phi^i\phi^j$; by spherical symmetry it can be reduced to a one dimensional integral.

Chapter 4

Harmonic Oscillator

28 The *simple harmonic oscillator* of classical mechanics is described by the solution of

$$\frac{d^2}{dt^2}q + \omega^2 q = 0$$

29 In quantum mechanics it has the hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}\omega^2 q^2$$

The spectrum is known to be

$$H|n >= \omega[n + \frac{1}{2}]|n >, \quad n = 0, 1, 2, 3 \cdots$$

30 We will recover this answer by path integral methods. It is enough to calculate the partition function

$$Z(\beta) = \text{ tr} e^{-H\beta} = \sum_{n=0}^{\infty} e^{-E_n\beta}.$$

31 From the discussion of Gaussian path integrals we see that it is the integral over all closed paths of period t :

$$Z(\beta) = \int_{q(0)=q(\beta)} \mathcal{D}q e^{-\frac{1}{2}\int_0^\beta [\dot{q}^2+\omega q^2]} d\tau$$

The answer is formally, $Z(t) = \sqrt{\det [2\pi A^{-1}]}$; integration by parts gives

$$A = -\frac{d^2}{d\tau^2} + \omega^2.$$

This is to be thought of as an operator on the space of period functions of period β .

32 The eigenvalues of A are, $\left[\frac{2\pi n}{\beta}\right]^2 + \omega^2$ for n an integer. So we need to make sense of the infinite product

$$Z(\beta) = \sqrt{\det \left[2\pi A^{-1}\right]} = \prod_{n=-\infty}^{\infty} \left\{ 2\pi \left(\left[\frac{2\pi n}{\beta}\right]^2 + \omega^2 \right)^{-1} \right\}^{\frac{1}{2}}.$$

32.1 We will rewrite it as a convergent product and an infnite product that does not involve ω :

$$\begin{aligned} Z(\beta) &= \frac{\sqrt{[2\pi]}}{\omega} \prod_{n=1}^{\infty} \left\{ 2\pi \left(\left[\frac{2\pi n}{\beta}\right]^2 + \omega^2 \right)^{-1} \right\} \\ &= \frac{\sqrt{[2\pi]}}{\omega} \prod_{n=1}^{\infty} \left[\frac{\beta}{\sqrt{[2\pi]} n}\right]^2 \prod_{n=1}^{\infty} \left\{ \frac{1}{1+\left[\frac{\omega\beta}{2\pi n}\right]^2} \right\} \end{aligned}$$

33 Now we recall that the Riemann zeta function

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}$$

is a convergent series in the right half plane $\Re s > 1$. By continuation we get a function analytic everywhere except for a simple pole at $s = 1$. In particular it is regular at $s = 0$. Moreover, it is well known that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log[2\pi]$.

34 Also, $\zeta'(s) = -\sum_1^\infty \frac{\log n}{n^s}$. So we can give a meaning to $\prod_1^\infty n^{-2} = e^{-2\sum_1^\infty \log n} = e^{2\zeta'(0)}$. Similarly $\prod_1^\infty z^2 = z^{2\zeta(0)}$. Thus we have

$$\prod_1^\infty n^{-2} = \frac{1}{2\pi}, \quad \prod_1^\infty z^2 = \frac{1}{z}.$$

34.1 This gives

$$\begin{aligned} Z(\beta) &= \frac{\sqrt{[2\pi]}}{\omega}\frac{\sqrt{[2\pi]}}{\beta}\frac{1}{2\pi}\frac{1}{\prod_{n=1}^\infty\left\{1+\left[\frac{\omega\beta}{2\pi n}\right]^2\right\}} \\ &= \frac{1}{\omega t\prod_{n=1}^\infty\left\{1+\left[\frac{\omega\beta}{2\pi n}\right]^2\right\}} \end{aligned}$$

35 Recall also the product formula

$$\sinh z = z\prod_{n=1}^\infty\left(1+\left[\frac{z}{n\pi}\right]^2\right).$$

Thus if we identify $z = \frac{\omega\beta}{2}$, we get

$$Z(\beta) = \frac{1}{2\sinh\left[\frac{\omega\beta}{2}\right]}.$$

36 Finally, we expand this in a geometric series:

$$Z(\beta) = \sum_{n=0}^\infty e^{-[n+\frac{1}{2}]\omega\beta}$$

giving the familiar answer for the spectrum of the harmonic oscillator:

$$E_n = \omega[n+\frac{1}{2}].$$

37 We see thus that the Riemann-zeta function gives the correct answer to the divergent products we encountered: *every single* numerical factor has come out right!.

38 Now let us look at the moments of the position variable. The simplest one is

$$< q(t)q(0) >= \int \mathcal{D}q e^{-\frac{1}{2}\int_{-\infty}^{\infty}[\dot{q}^2+\omega^2 q^2]d\tau} q(t)q(0).$$

We expect this to be the inverse of the operator $-\frac{d^2}{d\tau^2}+\omega^2$:

$$G(t,0) = \int \frac{dk}{2\pi}\frac{e^{ikt}}{k^2+\omega^2} = \frac{e^{-\omega|t|}}{2\omega}$$

All the other moments are given in terms of this one through Wick's theorem.

38.1 The physical interpretation is clearer if we note that the path integral on the right can be thought

$$< q(t)q(0) >= \lim_{T\to\infty}\frac{1}{N_T}\int h_T(q'',q)qh_t(q,q_0)q_0h_T(q_0,q')dqdq_0$$

where N_T is a normalization constant that ensures that $< 1 >= 1$. Now,

$$h_T(q,q') \to e^{-E_0T}\psi_0^*(q)\psi_0(q')$$

where ψ_0 is the ground state wavefunction and E_0 the ground state energy. (By adding a constant to the hamiltonian we can choose $E_0 = 0$.)Thus in the limit the rhs is independent of q', q'' :

$$< q(t)q(0) >= \int \psi_0^*(q)qh_t(q,q_0)q_0\psi(q_0) =< 0|\hat{q}e^{-\hat{H}t}\hat{q}|0 >$$

38.2 To get a physical interpretation we must analytically continue to real (physical) time:

$$\Delta(t) = \lim_{\epsilon\to 0^+} G(\epsilon+it) = \frac{e^{-i\omega t}}{2\omega}.$$

38.3 Looking back into the definition of $G(t)$, we see that

$$\Delta(t) = <0|e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t}\hat{q}|0>$$

Thus the moment of $q(.)$ is the (analytic continuation of the) ground state expectation value.

39 It is clear that the solution of the anharmonic oscillator with hamiltonian

$$H = p^2 + \frac{1}{2}\omega x^2 + \lambda x^4$$

wil invlove calculating path integrals that are not gaussian. The idea there is to expand in a series around the gaussian in powers of λ . Each term in this series can be calculated using Wicks theorem.

Chapter 5

Free Bosons are Harmonic Oscillators

40 If two quantum systems with Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are combined, the resultant system will, in general, have Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. If ψ_a and $\psi'_{a'}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{H}'$, the combined system will have as Hilbert space $\psi_a \otimes \psi'_{b'}$.

41 Thus, if we combine two *identical* systems, the Hilbert space is $\mathcal{H}\otimes\mathcal{H}$. This space carries a representation of the permutation group on two elements. We can split into a subspace of tensors that are invariant (*symmetric*) and another subspace of tensors that change sign (*anti-symmetric*) under permutation: $\mathcal{H}\otimes \mathcal{H} = S^2(\mathcal{H}) \oplus \Lambda^2\mathcal{H}$.

42 A pair of particles are indistinguishable if permuting them leaves the state invariant[1]; i.e., the vector in the combined Hilbert space representing their state changes by a phase under a permutation. Thus either it is unchanged or it changes sign.

43 There are thus two kinds of particles in nature: *bosons* whose state vector is unchanged by permutation and *fermions* whose state vector changes by a sign.

44 The axioms of quantum mechanics allow many additional possibilities when combining more than two partciles(e.g., higher dimensional representations of

[1] Recall that the state of a quantum system is described by a ray in a complex Hilbert space and not by a vector.

the permutation group.) But as yet these dont seem to appear in nature, at least as *fundamental* (indivisible) particles. So either the total Hilbert space of n particles is the space of rank n symmetric tensors (for bosons) or the space of rank n anti-symmetric tensors (for bosons).

45 Thus if ψ_a is an orthonormal basis for the single particle Hilbert space, an orthonormal basis for the n -particle Hilbert space is labelled by the *unordered* (but allowing for repetitions) labels $\{a_1 \cdots a_n\}$:

$$\sum_{\pi \in S_n} \frac{\psi_{a_{\pi(1)}} \otimes \psi_{a_{\pi(2)}} \otimes \cdots \psi_{a_{\pi(n)}} \otimes}{\sqrt{n!}}.$$

Here S_n is the permutation group on the n objects $1, 2, \cdots n$.

45.1 For example, for a pair of bosons, a basis is given by the symmetric products

$$\frac{\psi_{a_1} \otimes \psi_{a_2} + \psi_{a_2} \otimes \psi_{a_1}}{\sqrt{2}}$$

46 A basis vector above is dependent only on the number ν_a of times each label a appears in the list $a_1, \cdots a_r$: if it doesnt appear at all we set $\nu_a = 0$. Then, ν_a is the number of bosons that occupy the state a : the *occupation number*. Clearly $n = \sum_a \nu_a$. Given this sequence of occupation numbers ν_a , we can reconstruct the bosonic state:

$$|\nu >= \sum_{\pi \in S_n} \frac{\psi_{a_{\pi(1)}} \otimes \psi_{a_{\pi(2)}} \otimes \cdots \psi_{a_{\pi(n)}} \otimes}{\sqrt{n!}}$$

where $(a_1, \cdots a_r)$ is any sequence with the occupation numbers ν_a .

47 Of course a general state is obtained by taking linear combinations of the above basis elements. This is the Hilbert space of symmetric tensors of rank n .

48 It is convenient to combine all the different values of the number of particles (order of the tensors) into single Hilbert space called the *Bosonic Fock Space*:

$$S(\mathcal{H}) = \sum_{n=0}^{\infty} S^n(\mathcal{H}).$$

We identify $S^0(\mathcal{H})$ with C ; thsi one dimensional Hilbert space describes the *vacuum* or empty state. This allows linear combinations of elemenst where the number of particles may not be equal. Then $S^n(\mathcal{H})$ is the eigenspace of the *number operator* $\hat{N}$:

$$\hat{N}|\nu >= \left[\sum_a \nu_a\right] |\nu > .$$

49 Now suppose that the energy levels of the single particle system are known:

$$H\psi_a = \omega_a\psi_a.$$

if the bosons dont interact with each other the energy of the state with several particles occupying these states will be the sum of the individual energies. (If there is a repulsive interaction, the energy will be more than and if there is an attractive interaction it will be less.) Thus, for a system of free bosons, the multi-particle hamiltonian is an operator on the bosonic Fock space, whose eigenfunctions are the symmetric tensor products of those of H and whose eigenvalues are the sums of the individual energies:

$$\hat{H}|\nu >= \left[\sum_a \nu_a\omega_a\right] |\nu > .$$

Of course the energy of the empty or vacuum state is zero.

50 This is exactly the spectrum of a harmonic oscillator whose normal frequencies are ω_a , except that a constant must be added to the energy to make the energy of the ground state zero:

$$h = \sum_a \frac{1}{2}\left\{-\frac{\partial^2}{\partial q_a^2} + \omega_a q_a^2 - \omega_a^2\right\}$$

51 This suggests that we look for operators $a^{\dagger a}, a_a : S(\mathcal{H}) \to S(\mathcal{H})$ satisfying canonical commutation relations:

$$[a^{\dagger a}, a_b] = \delta^a_b, \quad [a_a, a_b] = [a^{\dagger a}, a^{\dagger b}] = 0$$

in terms of which

$$\hat{H} = \sum_a \omega_a a^{\dagger a} a_a, \quad \hat{N} = \sum_a a^{\dagger a} a_a.$$

52 Such operators are easy to find[2]:

$a^{\dagger a}|\nu>=\sqrt{[\nu_a+1]}|\cdots,\nu_a+1,\cdots>, \quad a_a|\nu>=\sqrt{[\nu_a]}|\cdots,\nu_a-1,\cdots>.$

In particular $a_a|0>=0$ for all a.

53 We interpret $a^{\dagger a}$ as creating a boson in the state a and a_a as annihilating one in that state. In there is no boson occupying that state, the annihilation operator will give zero; that is why it always gives zero on the vacuum state.

53.1 The simplest example is when a boson has just one state avaliable to it of energy ω; then the one particle Hilbert space is just $\mathcal{H}=C$. Even in this case $S(\mathcal{H})$ is infinite dimensional; a basis labelled by the occupation numbers is $|\nu>$, $\nu=0,1,2,3\cdots$. This corresponds to the simple harmonic oscillator with frequency ω.

54 The *Grand Canonical Partition Function* of a system of particles is defined to be

$$Z(\beta,\mu)=\ \mathrm{tr} e^{-\beta[\hat{H}-\mu\hat{N}]}.$$

The quantity β has the physical meaning of the reciprocal of *temperature*, and μ is the *chemical potential*. For free bosons with single particle energies, thsi can be easily calculated, since it becomes a product of geometric series:

$$Z(\beta,\mu)=\sum_{\nu_a=0,1,2,3\cdots} e^{-\beta\sum_a\nu_a[\omega_a-\mu]}=\prod_a\frac{1}{1-zq^{\omega_a}}.$$

Here, $q=e^{-\beta}$ and $z=e^{\beta\mu}$.

55 The *Gibbs Free energy* of a thermodynamical system is

$$F(\beta,\mu)=-\frac{1}{\beta}\log Z(\beta,\mu).$$

For free bosons,

$$F(\beta,\mu)=\frac{1}{\beta}\sum_a\log\left[1-zq^{\omega_a}\right]$$

[2] On the rhs we leave all the $\nu's$ unchanged except the one with label a

56 Let us apply this to a system of photons. Photons are particles of light; they are bosons. The state of a single photon is given by its momentum $\mathbf{k} \in R^3$ and its polarization $\sigma = \pm 1$. The energy of a photon is just equal to the magnitude of its momentum (because it is a particle of zero mass).

57 To get a discrete set of energies for a single particle (which we have been implicitly assuming so far) it is convenient to resort to a technical trick. We replace space by a cube of side R , then let R go to infinity at the end; moreover we assume periodic boundary conditions relating opposite faces of the cube. Then the allowed momenta are $\frac{2\pi}{R}\mathbf{m}$ where $\mathbf{m}$ is a vector whose components are integers. The Free energy becomes[3]

$$F(\beta,\mu) = \frac{2}{\beta} \sum_{\mathbf{k}\in\frac{2\pi}{R}Z^3} \log\left[1 - e^{-\beta[|\mathbf{k}|-\mu]}\right].$$

The free energy per unit volume has a definite limit as $R \to \infty$

$$f(\beta,\mu) = \lim_{R\to\infty} \frac{F(\beta,\mu)}{R^3} = \frac{2}{\beta} \int \frac{d^3k}{[2\pi]^3} \log\left[1 - e^{-\beta[|\mathbf{k}|-\mu]}\right].$$

58 Since the integrand only depends on the frequency (energy) of photons $\omega = |\mathbf{k}|$, we can write this as

$$f(\beta,\mu) = \frac{2}{\beta}\frac{1}{2\pi^2} \int_0^\infty \omega^2 d\omega \log\left[1 - e^{-\beta[\omega-\mu]}\right].$$

The

59 The average energy (per unit volume) is given by the derivative of $\beta f(\beta,\mu)$ with respect to β , (holding $\beta\mu$ fixed):

$$< E >= 2\frac{1}{2\pi^2} \int_0^\infty \omega^3 d\omega \frac{e^{-\beta[\omega-\mu]}}{1 - e^{-\beta[\omega-\mu]}} = \int_0^\infty \omega n(\omega) d\omega,$$

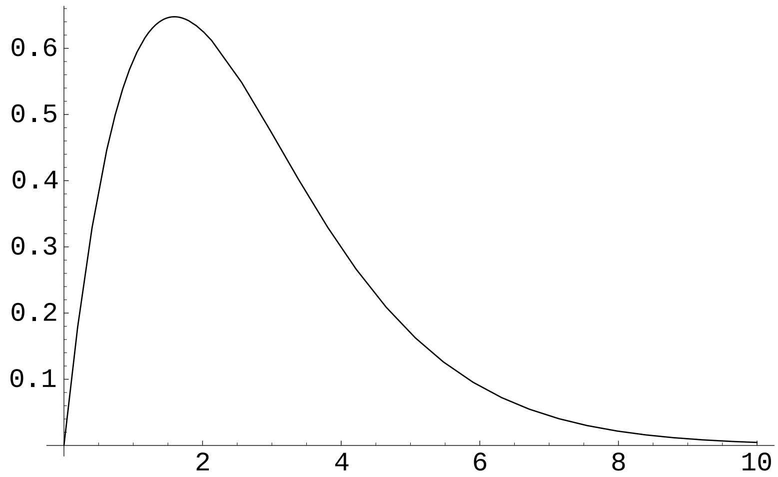
where

$$n(\omega) = \frac{1}{\pi^2} \frac{\omega^2}{e^{\beta[\omega-\mu]} - 1}$$

is the number density of photons as a function of frequency.

[3] The factor of two counts the two polarization states for a given momentum.

60 The simplest case is $\mu = 0$, which means that we allow emission and abosrption of photons with no cost of energy. The number density of photons in this case is the celebrated Planck distribution (see figure below). Planck originally derived it by fitting the observed density of photons at a finite temperature. Bose's derivation of this result based on the idea of indistinguishability was one of the early triumphs of quantum physics.



61 Einstein, in a note attached to Bose's paper, discovered that this system of free bosons has a phase transition at a low temperature. This phenomenon of Bose-Einstein condensation was very surprising and a direct verification has only been possible in the 1990's. There has been a revival of interest in the theory of nearly free systems of bosons as a result of recent experimental advances on cool (nanokelvin) gases of atoms.

62 The space of symmetric tensors over a vector space can be identified with the space of polynomials (see notes on PHY 405, *Differential Geometry for Physics Graduate Students*.

Chapter 6

Free Fermions

63 Pauli discovered that a simple explanation of the periodic table of elements could be given at most one electron can occupy each atomic state (including spin in the counting of states). This *Pauli exlusion principle* was a key step in the reduction all of chemistry to physics: a basic achievement of the twentieth century .

64 If the wavefunction of identical particles is completely antisymmetric, at most one can occupy each state. Particles with this property are called *Fermions*. If electrons are fermions, *Pauli exclusion principle*. There are many other examples of fermions as well; a key fact of nature is that any particle of half integer spin (whose wavefunction changes sign under rotation through 2π) is a fermion.

65 Thus for a fermion, the Hilbert space of states of r particles is the exterior product $\Lambda^r(\mathcal{H})$ of the one-particle Hilbert space. The *Fermionic Fock space* is the set of all multi-particle states, $\Lambda(\mathcal{H}) = \oplus_{n=0}^{\infty}\Lambda^n(\mathcal{H})$. If ψ_a is an orthonormal basis of a single particle, a basis for a basis for $\Lambda^r(\mathcal{H})$ is

$$\psi_{a_1} \wedge \cdots \psi_{a_r} = \sum_{\pi \in S_n} \mathrm{sgn}\ (\pi) \frac{\psi_{a_{\pi(1)}} \otimes \psi_{a_{\pi(2)}} \otimes \cdots \psi_{a_{\pi(n)}} \otimes}{\sqrt{n!}}.$$

Here S_n is the permutation group on the n objects $1, 2, \cdots n$. Also, $\mathrm{sgn}\ (\pi$) is the sign of the permutation π : if π is the product of an even (respectively odd) number of pairwise permutations, it is one (respectively -1).

66 Since the state above is antisymmetric, the indices cannot be equal; moreover, we can bring them to the increasing order by some permutation:

$$a_1 < a_2 < \cdots a_n.$$

As in the bosonic case, we can relabel this list by the number ν_a (*occupation number*) of times each index a appears: each ν_a can take values $0, 1$ independent of the others and

$$\sum_a \nu_a = n.$$

The orthonormal basis will thus be denoted by $|\nu>$ in the occupation number description. We can define the *number operator* by

$$\hat{N}|\nu> = \sum_a \nu_a |\nu> .$$

67 Suppose the single particle energy levels are ω_a with ψ_a as the eigenstates. The hamiltonian of the multiparticle system will have $|\nu>$ as the eigenstates:

$$\hat{H}_F = \sum_a \omega_a \nu_a |\nu> .$$

68 The *Grand Canonical Partition function* of a system of free fermions can now be calculated:

$$Z(q, z) = \operatorname{tr} q^{H_F} z^{\hat{N}} = \sum_{\nu_a = 0,1} q^{\sum_a \nu_a \omega_a} z^{\nu_a} = \prod_a [1 + z q^{\omega_a}] .$$

Here $q = e^{-\beta}$, β being inverse temperature as usual; the *fugacity* $z = e^{\beta\mu}$, where μ is the *chemical potential*.

69 As in the bosonic case, we can now define the *creation* and *annihilation* operators $c_a, c^{\dagger a}$; but here, they will toggle the values $0, 1$ of ν_a . Thus we must expect that $c_a^2 = 0 = \left[c^{\dagger a}\right]^2$. A way of realizing this is to mimic the case of bosons, by postulating the *Canonical Anti-Commutation Relations*:

$$[c_a, c^{\dagger b}]_+ = \delta_i^j, \quad [c_b, c_b]_+ = [c^{\dagger a}, c^{\dagger b}]_+ = [c_a, c_b]_+ = 0$$

where the *anti-commutator* $[A, B]_+$ of a pair of operators is defined to be

$$[A, B]_+ = AB + BA.$$

70 There is a realization of this algebra on the fermionic Fock space.The empty state satisfies

$$c_a|0>=0;$$

The state $|\nu_a>$ is obtained from this empty state by

$$|\nu_a>=c^{\dagger a_1}\cdots c^{\dagger a_n}|0>$$

where $a_1\cdots a_n$ is the sequence of indices for which $\nu_a=1$. We define $\hat{N}_a=c^{\dagger a}c_a$ as the *number operator* for the label a , since

$$c^{\dagger a}c_a>|\nu>=\nu_a|\nu>.$$

Thus we have

$$\hat{H}_F=\sum_a\omega_a c^{\dagger a}c_a,\quad \hat{N}_F=\sum_a c^{\dagger a}c_a.$$

70.1 In the simplest case where there is only one single particle state, we have the *fermionic harmonic oscillator*

$$c^2=c^{\dagger a}=0, cc^\dagger+c^\dagger c=1$$

and

$$\hat{H}_F=\omega c^\dagger c.$$

There are two states

$$c|0>=0,\quad c^\dagger|0>=|1>,\quad \hat{H}_F|0>=0,\quad \hat{H}_F|1>=\omega|1>.$$

71 The bosonic mutliparticle states can be thought of as polynomials on a variable, with the creation operator being the multiplication operator and the annihilation operator being differentiation. This idea can be generalized to the fermionic case, but the variables must satisfy the *exterior algebra* (also called *Grassmann algebra*):

$$\theta^a\theta^b+\theta^b\theta^a=0.$$

In other words we consider the associative algebra generated by the θ^a modulo the ideal generated by the above relations. A general polynomial in these Grassmann variables a linear combination

$$\sum_{n=0}^{\dim \mathcal{H}} \sum_{a_1 \cdots a_n} \xi_{a_1 \cdots a_n} \frac{\theta^{a_1} \cdots \theta^{a_n}}{n!}.$$

The relations on the Grassmann variables force the coefficients to be antisymmetric tensors.

72 As a vector space, this space of polynomials is the same as the Fermionic Fock space $\Lambda(\mathcal{H})$: they are just different ways of looking the space of anti-symmetric tensors. The state $|\nu>$ is the monomial

$$\theta^{a_1} \cdots \theta^{a_n}$$

where $a_1 \cdots a_n$ is the sequence of indices for which $\nu_a = 1$. The empty state is the constant 1 ; the creation operator $c^{\dagger a}$ is the multiplication by the Grassmann variable θ^a . We now define the derivative operator to correspond to the annihilation operator:

$$\frac{\partial}{\partial \theta_a} \theta^{a_1} \cdots \theta^{a_n} = \sum_{k=1}^{n} [-1]^{k-1} \delta_a^{a_k} \theta^{a_1} \cdots \widehat{\theta^{a_k}} \theta^{a_n}$$

Here $\widehat{\theta^{a_k}}$ means we must omit that entry in the product.

72.1 In the simplest case where $\dim \mathcal{H} = 1$, the most general polynomial is

$$\xi(\theta) = \xi_0 + \xi_1 \theta$$

since $\theta^2 = 0$. Here ξ_0, ξ_1 are complex numbers. Also, $\theta \xi(\theta) = \xi_0 \theta$, $\frac{\partial}{\partial \theta} \xi(\theta) = \xi_1$. We then have

$$\theta^2 = 0 = \frac{\partial^2}{\partial \theta^2}, \quad \frac{\partial}{\partial \theta} \theta + \theta \frac{\partial}{\partial \theta} = 1$$

giving a representation of the canonical anti-commutation relations.

72.2 We might wonder if there is a notion of integration for these functions of a Grassmann variable. A integral would be a map from the space of functions to the complex numbers satisfying the axioms of linearity and translation invariance:

$$\int [a\xi(\theta)+b\eta(\theta)]d\theta = a\int \xi(\theta)d\theta + b\int \eta(\theta)d\theta, \quad \int d\theta\xi(\theta+a) = \int d\theta\xi(\theta).$$

Here a and b are complex numbers and $\xi(\theta+a)=\xi_0+\xi\theta+\xi a$. These condtions imply (upto a multiplicative constant)

$$\int d\theta 1 = 0, \quad \int d\theta\theta = 1$$

or that the integral simply picks out the top order term:

$$\int d\theta\xi(\theta) = \xi_1.$$

73 We can now define a complex conjugation of the functions Grassmann variables:

$$\xi^*(\theta) = \xi_0^* + \xi_1^*\theta^*$$

We regard θ^* as another Grassmann variable indepndent of θ and anti-commuting with it:

$$\theta^2 = 0 = \theta^{*2}, \theta\theta^* + \theta^*\theta = 0.$$

The previous functions of θ alone are like analytic functions. A general function, not necessarily analytic, would be

$$\xi(\theta,\theta^*) = \xi_0 + \xi_1\theta + \xi^1\theta^* + \xi_1^1\theta^*\theta$$

The integral of such a function is again the top order form:

$$\int d\theta d\theta^*\xi(\theta,\theta^*) = \xi_1^1$$

This is determined (upto a constant) by the conditions of linearity and translation invariance in each variable. The inner product of a pair of fermionic statescan now be thought of as the integral:

$$<\xi,\eta> = \xi_0^*\eta_0 + \xi_1^*\eta_1 = \int d\theta d\theta^* e^{\theta\theta^*}\xi^*(\theta^*)\eta(\theta).$$

Here we use $e^{\theta^*\theta} = 1 = \theta\theta^*$.

74 Suppose $\dim \mathcal{H}$ is finite, equal to N. Then we define the *integral* of a function of the Grassmann variables θ_a to be just the top order term in their expansion:

$$\int d\theta \xi(\theta) = \xi_{12\cdots N}.$$

If we have functions of a set of Grassmann variables and their complex conjugates satisftying

$$\theta^a\theta^b + \theta^b\theta^a = \theta^*_a\theta^*_b + \theta^*_b\theta^*_a = \theta^a\theta^*_b + \theta^*_b\theta^a = 0$$

we define again the integral to be the top order term which is now an antisymmetric tensor of type (N, N).

$$\int d\theta d\theta^* \xi(\theta, \theta^*) = \xi^{1\cdots N}_{1\cdots N}.$$

Thus the states of a multi-fermionic system can be thought of as analytic functions of a set of Grassmann variables. With this the inner product of a pair of anti-symmetric tensors can be thought as an integral:

$$<\xi, \eta> = \sum_n \frac{1}{n!}\xi^*_{i_1\cdots i_n}\eta^{i_1\cdots i_n} = \int d\theta d\theta^* e^{\theta^*\cdot\theta}\xi^*(\theta^*)\eta(\theta).$$

75 We will soon develop a theory of Gaussian integrals over Grassmannian variables; upon extension to the infinite dimensional case it will give a path integral formulation for free fermions.

Chapter 7

Fermionic Integrals

76 The determinant of an anti-symmetric matrix of odd dimension is zero. The determinant of an anti-symmetric matrix of even dimension is the square of a polynomial in its matrix elements. This polynomial is called the Pfaffian. The square root of the determinant of a symmetric matrix appears in evaluating a Gaussian integral over real or complex variables; the Pfaffian of an antisymmetric matrix appears in the analogous Gaussian integral over Grassmann variables.

77 Let us begin by recalling the notion of wedge products of anti-symemtric tensors; we will need only the wedge product of a tensor with itself [1]. Given an $2N \times 2N$ anti-symmetric matrix Ω , define the *wedge powers* of an antisymmetric tensor Ω^r

$$[\Omega^r]_{i_1 \cdots i_{2r}} = \sum \text{sgn}\ (i_1 \cdots i_{2r}, j_1 \cdots j_{2r})\Omega_{j_1 j_2} \cdots \Omega_{i_{2r-1} i_{2r}}.$$

Here, the sum is over all the partitions of the set $\{i_1 \cdots i_n\}$ as unions of pairs $\{j_1, j_2\} \cup \cdots \cup \{j_{2r-1}, j_{2r}\}$; also, $\text{sgn}\ (i_1 \cdots i_{2r}, j_1 \cdots j_{2r})$ is the sign of the permutation taking the i 's to the j 's. The *Pfaffian* is then the wedge product to the highest power possible:

$$\text{Pf}\Omega = \left[\Omega^N\right]_{123 \cdots 2N}.$$

78 To see the connection with determinants recall that

$$\det \Omega = \sum \text{sgn}\ (j_1 \cdots j_{2N})\Omega_{1 j_1} \cdots \Omega_{2N j_{2N}}$$

[1]For a modern perspective, (details are not needed here) see D. E. Knuth, *Overlapping Pfaffians* Electronic Journal of Combinatorics 3 (2) (1996).

where $\mathrm{sgn}\,(j_1\cdots j_{2N})$ is the sign of the permutation relating $12\cdots 2N$ to $j_1\cdots j_{2N}$; also we sum over all such permutations. Then we can see that $\det\Omega = [\mathrm{Pf}\ \Omega]^2$.

79 Another useful formula is the recursion relation

$$[\Omega^r]_{i_1\cdots i_{2r}} = \sum_{k=2}^{2r} \Omega_{i_1 i_k} \left[\Omega^{r-1}\right]_{i_{k+1}\cdots i_{2r} i_1 i_2\cdots i_{k-1}}.$$

This allows a calculation of the Pfaffian by a rule analogous to the Kramers rule for determinants.

80 The Pfaffian of an anti-symmetric matrix of odd order is zero.

80.1 We see now that $\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$; if Ω is a four by four matrix with $\Omega_{0i} = E_i, \Omega_{12} = B_3$ etc., $\mathrm{Pf}\ \Omega = \mathbf{E}\cdot\mathbf{B}$.

81 We define the integral of a function of a single Grassmann variable by

$$\int d\theta 1 = 0, \quad \int d\theta\theta = c.$$

The constant c will be fixed later. Upto this constants, these are fixed by the psotulates of linearity and translation invariance.

82 For functions of N Grassmann variables we define

$$\int d\theta\theta^N\theta^{N-1}\cdots\theta^1 = c^N;$$

and the integral of any lower order monomial to zero. This integral can be thought of as a product of N integrals over single variables. We are to think of $d\theta$ as $d\theta^1 d\theta^2\cdots d\theta^N$, which explains the choice of ordering of the factors.

83 Notice right away that fermionic integrals have the opposite scaling behavior to bosonic integrals:

$$\int d\theta f(a\theta) = a^N \int d\theta f(\theta).$$

Indeed integration and differentiation are the same operation (upto a constant).

84 With this definition,

$$\int d\theta e^{-\frac{1}{2}\Omega_{ij}\theta^i\theta^j} = c^N \text{Pf } \Omega$$

where we integrate over N Grassmannian variables.

84.1 It is clear that expanding the exponential will give a polynomial; moreover the top power is all that matters to the integral. Thus the integral is proportional to the Pfaffian; it is the only completely anti-symmetric tensor of order $2N$ we can get from Ω . If we choose $\Omega = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $\text{Pf } \Omega = 1$. Also the integral factorizes into the product of two dimensional integrals. In this case the integral evaluates to

$$\int d\theta^1 d\theta^2 e^{-\theta^1\theta^2} = -\int d\theta^1 d\theta^2 \theta^1 \theta^2 = c^2.$$

85 Thus we have a close analogy with the bosonic Gaussian integral

$$\int dx e^{-\frac{1}{2}A_{ij}x^i x^j} = \sqrt{\det}\left[2\pi A^{-1}\right].$$

The fermionic integrals scale oppositely to the bosonic integrals; that is why we have the square root of the determinant in the numerator. We can make the analogy even closer by the definition

$$c = \frac{1}{\sqrt{[2\pi]}}.$$

86 Then we have

$$\int d\theta e^{-\frac{1}{2}\Omega_{ij}\theta^i\theta^j} = \text{Pf }\left(\frac{\Omega}{2\pi}\right).$$

In particular the integral vanishes for an odd number of variables.

87 Let Ω be a non-degenerate (i.e., with non-vanishing Pfaffian) anti-symmetric matrix. We define the *Gaussian expectation value* of a function by

$$< f > = \frac{\int d\theta e^{-\frac{1}{2}\Omega_{ij}\theta^i\theta^j} f(\theta)}{\int d\theta e^{-\frac{1}{2}\Omega_{ij}\theta^i\theta^j}}$$

88 The moments of a fermionic gaussian random variable are now given by an analogue of Wicks theorem:

$$< \theta^{i_1} \cdots \theta^{i_{2r}} >= \sum \text{ sgn } (i_1 \cdots i_{2r}, j_1 \cdots j_{2r}) S^{j_1 j_2} \cdots S^{i_{2r-1} i_{2r}}.$$

Here S is the inverse of the matrix Ω . Also, the sum is over all the partitions of the set $\{i_1 \cdots i_n\}$ as unions of pairs $\{j_1, j_2\} \cup \cdots \cup \{j_{2r-1}, j_{2r}\}$; also, $\text{sgn } (i_1 \cdots i_{2r}, j_1 \cdots j_{2r})$ is the sign of the permutation taking the i 's to the j 's.

89 In other words, the moment is just the wedge power of the *covariance matrix* $S = \Omega^{-1}$.

$$< \theta^{i_1} \cdots \theta^{i_{2r}} >= [S^r]^{i_1 \cdots i_{2r}} .$$

89.1 Of course, the expectation value will vanish for odd orders. If the number of Grassmannian variables is odd there is no non-degenerate anti-symmetric tensor Ω so the expectation value is not defined. S is the covariance matrix because

$$< \theta^i \theta^j >= S^{ij}.$$

Chapter 8

Lorentz Invariant Equations

90 The *Euclidean group* consists of transformations that leave the square of the distance $(x-y)^2 = \sum_i^n (x^i - y^i)^2$ between two points in R^n invariant. Such a transformation is labelled by a matrix and a vector (R, a) , $x \mapsto Rx + a$. a can be any vector in R^n , but R must be orthogonal: $R^T R = 1$. The compostion law of transformations is $(R, a)(R', a') = (RR', a + Ra')$. The subgroup $(R, 0)$ consists of *rotations* in some plane (when $\det R = 1$), and reflections when $det = -1$. The Euclidean group is the semi-direct product of the group of orthogonal matrices with the group of translations.

91 Orthogonal transformations infinitesimally close the identity, $R = 1 + \theta$ describe infinitesimal rotations in some plane. There are $\frac{n(n-1)}{2}$ independent such transformations. A basis is given by the matrices e_{ij} for $i < j$:

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Here $(e_{ij})_{kl}$ means the kl matrix element of e_{ij} . They satisfy the commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - [i \leftrightarrow j, k \leftrightarrow l] .$$

These are the relations of the orthogonal Lie algebra. This can be supplemented with the relations of the infinitesimal translations:

$$[e_{ij}, v_k] = \delta_{jk} v_i - \delta_{ik} v_j, \quad [v_i, v_j] = 0.$$

The case of most interest is of course $n = 3$; but we will describe the general case as it will make the step to the relativistic theory easier.

92 A free particle in quantum mechanics has as its basic symmetry the euclidean group. The Hilbert space of a free particle must carry a representation of the above commutation relations. The simplest example[1] would be that $\mathcal{H} = L^2(R^3, C)$,

$$t_i \mapsto P_i = -i\frac{\partial}{\partial x^i}, \quad e_{ij} \mapsto L_{ij} = x_i P_j - x_j P_i.$$

The translations are generated by the momentum operator and rotations by the orbital angular momentum operator.

93 A more intricate possibility is that the wave functions take values in some finite dimensional vector space C^N , so that $L^2(R^3, C^N)$. The representation is

$$t_i \mapsto P_i = -i\frac{\partial}{\partial x^i}, \quad e_{ij} \mapsto J_{ij} = x_i P_j - x_j P_i + \Sigma_{ij}$$

where Σ_{ij} are $N \times N$ constant matrices that by themselves satisfy the commutation relations of the rotation algebra. In ths situation, a particle could have non-zero angular momentum even when it is at rest; Σ_{ij} represent this *spin* angular momentum.

94 The simplest example of spin matrices is $\Sigma_{ij} = \frac{1}{2\sqrt{[-1]}}[\sigma_i, \sigma_j]$ and the Pauli matrices σ_i are defined by the relations

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}.$$

In the case of most physical interest, three dimensional Euclidean space, there is a standard choice of such matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This describes particles of spin half: the minimum value of angular momentum is $-\frac{1}{2}$.

95 These ideas generalize naturally to the relativistic case. We must consider four dimensional Minkowslki space $R^{1,3} = \{(x^0, x^1, x^2, x^3)\}$ in which the square

[1] We use physicists convention that the operators that represent a symmetry are self-adjoint; this means that the commutation relations are satisfied upto a factor of $-i$.

of the distance of a pair of points is $(x-y)\cdot(x-y) = (x^0-y^0)^2 - \sum_{i=1}^{3}(x^i-y^i)^2$. Translations are again a symmetry; the linear transformations $x \mapsto \Lambda x$ are now symmetries if $\Lambda^T \eta \Lambda = \eta$. (Here $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1_{3\times 3} \end{pmatrix}$ is the *Minkowski metric*.) The group of such matrices is the *Lorentz group*. The full symmetry (the *Poincaré group*) is the semi-direct product of the Lorentz group with the translations.

96 The infinitesimal Lorentz transformations $\Lambda = 1+\lambda$, are matrices satisfying $\lambda^T\eta + \eta\lambda = 0$; in other words $\eta\lambda$ is anti-symmetric.A basis is given by the elementary transformations $e_{\mu\nu}x^\rho = \delta^\rho_\nu \eta_{\mu\sigma} x^\sigma - [\mu \leftrightarrow \nu]$. They satisfy the commutation relations of the *Lorentz Lie algebra*:

$$[e_{\mu\nu}, e_{\rho\sigma}] = \eta_{\nu\rho} e_{\mu\sigma} - [\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma] \, .$$

If we add in translations we get the *Poincaré Lie algebra*

$$[e_{\mu\nu}, v_\sigma] = \eta_{\nu\sigma} v_\mu - \eta_{\mu\sigma} v_\nu, \quad [v_\mu, v_\nu] = 0.$$

97 A free particle in relativistic quantum mechanics is decsribed by a representation of this Lie algebra. The most obvious representation is on the space of complex valued functions on space time:

$$v_\mu \mapsto P_\mu = -i\frac{\partial}{\partial x^\mu}, e_{\mu\nu} \mapsto L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu.$$

This describes spin zero particles as is seen by restricting to the subalgebra of rotations.

98 Now, P_μ is a four-vector that represents energy and momentum. But they are not independent of each other. For a particle of mass m we have the classical relation . Quantum mechanically, the wavefunction of a single free spinless particle must satisfy the partial differential equation

$$\left[\eta^{\mu\nu} P_\mu P_\nu - m^2\right] \phi = 0.$$

This is the Klein-Gordon equation.

98.1 This equation can be solved by Fourier analysis:

$$\phi(x) = \int \tilde{\phi}(p) e^{ip\cdot x} \frac{d^4p}{[2\pi]^4}.$$

We get

$$[\eta^{\mu\nu} p_\mu p_\nu - m^2]\tilde{\phi}(p) = 0.$$

Thus the wavefunction vanishes except when the energy-momentum relation is satisfied. This is a two-sheeted hyperboloid (*mass shell* or *mass hyperboloid*); the space of complex valued functions on this hyperboloid is isomorphic to the space of solutions of the Kelin-Gordon equation. This space carries a representation of the Poincaré group, as the Klein-Gordon equation is invariant. In fact this is an irreducible representation.

99 We can look for other realizations that describe particles with spin. By analogy with the Pauli matrices we introduce the *Dirac matrices*:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}.$$

A standard choice (all others are related to this one by a similarity transformation $\gamma_\mu \mapsto S\gamma_\mu S^{-1}$) is

$$\gamma_0 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

100 It can be verified that the operators

$$P_\mu = -i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \Sigma_{\mu\nu}$$

(where $\Sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$) satisfy the commutation relations for the Poincaré algebra.

101 We still need to impose a partial differential equation analogous to the Kelin-Gordon equation; indeed the Klein-Gordon equation must be implied by it as the energy-momentum relation holds. The key step is to note that the Dirac matrices γ_μ transform as vectors under $\Sigma_{\mu\nu}$ and hence under $J_{\mu\nu}$.

102 Thus the first order partial differential equation (*Dirac equation*)

$$[\gamma^\mu P_\mu + m]\,\psi = 0$$

is invariant under the Lorentz transformations. It implies the Klein-Gordon equation (and hence the correct energy-momentum relation) since

$$[\gamma^\mu P_\mu - m]\,[\gamma^\mu P_\mu + m] = \eta^{\mu\nu} P_\mu P_\nu - m^2.$$

Here we use the defining relations of the Dirac matrices and the fact that $[P_\mu, P_\nu] = 0$.

103 Again we can solve the equation by Fourier analysis:

$$\psi(x) = \int e^{ip\cdot x}\tilde{\psi}(p)\frac{d^4p}{(2\pi)^4} \Rightarrow [\gamma^\mu p_\mu + m]\tilde{\psi}(p) = 0.$$

For each momentum on the mass hyperboloid, there will be two independent solutions.