PHY 511 Field Theory

Problem set 1 Fall 2000

Due Oct 4 2000

Please put your solutions in Abhishek Agarwal's mailbox by the end of the day on Wednesday.

1. (i) Define the 'fermionic oscillator' by the canonical anticommutatir relations

$$cc^\dagger + c^\dagger c = 1, \quad c^2 = c^{\dagger 2} = 0$$

and the hamiltonian $H = \omega c^\dagger c$. Find the energy levels, and the eigenstates.

(ii) Let θ be a Grassmann variable satisgying the relation $\theta^2 = 0$. Find a representation of the above operators in terms of θ and the derivative with respect to θ . What are the eigenfunctions of the hamiltonian?

2. Determine the sixth moment, $< \phi^{i_1} \cdots \phi^{i_6} >$ for a Gaussian random variable ϕ^i with zero mean and covariance matrix C .

3. (i) Evaluate the integral $I(\lambda) = \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}\phi^2 - \lambda\phi^4}$ in terms of standard functions of mathematical physics (e.g., Bessel functions.)

(ii) Expand $I(\lambda)$ in powers of λ to order λ^4 .

(iii) Determine the behavior of the coefficients of the this power series for large order. Is it a convergent series?

4[1]. Define the *Riemann zeta function* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

(i) Show that this series converges uniformly in the right half plane $\mathrm{Re}\, s > 1$; and that it defines a function analytic everywhere except for a simple pole at $s = 1$.

(ii) Find the values $\zeta(0), \zeta'(0)$.

(iii) Use this to evaluate the infinite sum $\sum_{n=1}^{\infty} 1$ and the infinite product $\prod_{n=1}^{\infty} n$.

[1] This problem is harder than the others

PHY 511 Field Theory

Problem set 2 Fall 2000

Due Oct 30 2000

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5. (i) Obtain the commutation relations of infinitesimal Lorentz transformations. Find a representation for them in terms of 4×4 Dirac matrices satisfying $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$.

(ii) Show then that the Dirac equation $[-i\eta^{\mu\nu}\gamma_\mu \partial_\nu + m]\psi = 0$ is invariant under Lorentz tranformations.

6. Show that $\bar{\psi}\gamma_\mu \psi$, where $\bar{\psi} = \psi^\dagger \gamma_0$ is a Lorentz vector; and that it is conserved if the Dirac equation is satisfied.

7. (i) Show that the determinant of an antisymmetric matrix of odd dimension is zero; and that the determinant of an anti-symmetric matrix is the square of a polynomial (the *Pfaffian*)in its matrix elements.

(ii) Using the rules for Grassmannian integration, express the integral over $2n$ Grassmann variables $\int d\theta e^{\frac{1}{2}\theta^T B \theta}$ in terms of the Pfaffian of the antisymmetric matrix B .

8. (i) Evaluate the Pfaffian of the operator $\frac{d}{d\tau} + i\omega$ over the space of functions that satisfy the antiperiodic boundary conditions $\psi(\beta) = -\psi(0)$. Use the zeta function method to deal with the infinite products.

(ii) Compare with the solution of the fermionic oscillator to get a path integral formulation for it.

PHY 511 Field Theory

Problem set 3 Fall 2000

Due Dec 4 2000

9. Evaluate the integral

$$f(n,s,\sigma) = \int \frac{d^nk}{(2\pi)^n} \frac{1}{(k^2+\sigma)^s}$$

as an analytic function of the complex variables n, s, σ .

(i) Where are the singularities as a function of s when $n = 1, 2, 3, \cdots$ and $\sigma > 0$?.

(ii) Find the logarithm of the Determimant of the operator $-\nabla^2 + \sigma$ in n -dimensional Euclidean space when $\sigma > 0$ is a constant by using an appropriate analytic continuation of the above function.

(iii) Find the one-loop beta function for the renormalization of $\lambda\phi^4$ theory using the above computation.

10. Consider the theory where the fermions ψ are coupled to scalar fields :

$$S = \int \left(\frac{1}{2}[\nabla\phi]^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + \bar{\psi}[\gamma \cdot + g\phi]\psi \right) dx.$$

(i) Find the one-loop effective potential for the scalar field including the contribution of the Fermi field.

(ii) Find the beta vector field of the three coupling constants m, g, λ at one loop.

11[2]. Consider the scalar field theory with the action

$$S = \int \left([\nabla\phi]^2 + W(\phi) \right) d^nx$$

where the function W is not necessarily a polynomial.

(i) Find the beta-vector field in first order of perturbation theory, where we treat the function $W(\phi)$ as a perturbation to the free massless scalar theory.

[2]This is a hard problem, for the more advanced students

(ii) Derive a partial differential equation that describes the evolution of the effective potential under the renotmalization group for $n = 1, 2, 3, 4$.

(iii) What are the renormalizable polynomial interactions in these dimensions?

(iv) Find a renormalizable interaction in $n = 2$ which is not a polynomial.

(v) Same in the case $n = 3, 4$.