12. Moving Charges

S. G. Rajeev

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How would we find the magnetic field produced by a wire that is not straight? We could imagine breaking it up into smaller bits and adding up the field producing by each bit. This is the idea of the Biot-Savart law. This is similar to the way we think of charge distributions as made of point charges. Instead of a point charge we have a piece of wire of very small length.

1 Field of a thin wire

Consider a part of a wire of infinitesimally small length $d\mathbf{l}$. The wire carries a current I. The direction of the vector $d\mathbf{l}$ is the direction of the current at that point on the wire. The magnetic field produced is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\hat{\mathbf{r}}}{r^2}.$$

This is in like the Coulomb's law for electric fields. Instead of $\frac{1}{\epsilon_0}$ we have μ_0. Instead of an electric charge q we have the current. But the current flows in some direction, which is given by the infinitesimal tangent vector to the wire $d\mathbf{l}$. The magnetic field produced is perpendcular both to the current and to the unit vector $\hat{\mathbf{r}}$ connecting the piece of wire to the point of observation. The total magnetic field at the position $\mathbf{r}$ is given by the integal alone the wire:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\frac{d\mathbf{l}\times\hat{\mathbf{r}}}{r^2}$$

This is known as the Biot-Savart law.

2 Current Density

Sometimes we may need to take into account the thickness of the wire. Or the charges might be moving in a large region of space. For example, the Sun is so hot that the atoms have been stripped of their electrons and they are moving about independently. In such a plasma there will be a current spread out over a large volume. Same thing happens when an electric current passes through an electrolyte solution.

The current that crosses a given a plane surface divided by its area is called the current density. Current density is a vector $\mathbf{j}$ pointed along the normal to the surface. The magnetic field produced by a small region of space dV centered at some point $\mathbf{r}'$is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}(\mathbf{r}') \times \widehat{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}dV.$$

The total magnetic field is given by integrating over the colume

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}') \times \widehat{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}dV.$$

This is similar to a formula we derived for the electric field as an integral over charge density:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')\widehat{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}dV.$$

Charge denisty ρ is a scalar while current density $\mathbf{j}$ is a vector. So in the case of the magnetic field we must take the product of two vectors $\mathbf{j}$ and the unite vector.

3 Magnetic Field of a Moving Charge

We saw that a point charge Q at rest at the origin produces an electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{\mathbf{r}}}{r^2}.$$

It it is sitting at the point $\mathbf{r}'$ we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q\widehat{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}.$$

A charge that is moving will in addition also produce a magnetic field: the movement of the charge is a current, which is a source of magnetic field. From the formula above we can deduce that the magnetic field produced is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q\mathbf{v} \times \widehat{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}.$$

where $\mathbf{v}$ is the velocity of the charge. Thus the electric and magnetic fields produced by a single particle are related:

$$\mathbf{B} = \mu_0\epsilon_0\mathbf{v} \times \mathbf{E}.$$

This quantity $\epsilon_0\mu_0$ has the dimensions of one over velocity squared.

What is this velocity?
If we calculate

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} 8.854187817 \times 10^{-12}}} = \frac{10^{10}}{\sqrt{4\pi \times 0.88541878}}$$

$$= 299,792,458ms^{-1}.$$

This is the speed of light!

Thus the electric and magnetic fields produced by a point charge moving at a constant velocity $\mathbf{v}$ which is small compared to c are related by

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E}.$$

Maxwell, who noticed this first, suspected that light has something to do with electricity and magnetism. This was a big surprise in its day. He later went on to explain the origin of light from electricity . It turns out that *accelerating* charged particles will emit light. We will study Maxwell's theory later on in the course.

4 Speed of Light in Media

The speed of light in the vacuum is a constant of nature. But in most transparent media (like water) the speed of light is smaller than in the vacuum. This because the permittivity ϵ of the medium is larger than that of vacuum: it has a dielectric constant greater than one. The permeability μ can be larger too, but most such substances (like iron) are not transparent: they absorb light.

Particles can then travel faster than light inside such media. Then they will emit light: a phenomenon known as Cerenkov radiation. It is similar to the 'sonic boom' of sound emitted by airplanes travelling than the speed of sound.

Cerenkov radiation has found some applications to biology. You can inject minute quantities of an isotope of Phosporus (atomic weight 32) into an organism. This isotope emits electrons at a speed close to c. They can be detected using the Cerenkov radiation they emit in the water in the tissue. So we can see where phosphorus is being absorbed.

But Cerenkov radiation is mostly used by physicists to see fast moving charged particles. For example, there is a large array of detectors that look for the Cerenkov radiation emitted by high energy cosmic rays as they pass through Antartic ice. We hope to understand the source of such such rays. They are believed to be emitted when a star falls into a blackhole.