S11 PHY114 Problem Set 3

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1. The electric flux through the surface of a box whose sides are $8.1 \times 11.5 \times 12.2$ cm is $2000 N m^2 C^{-1}$. (a) What is the total electric charge inside the box? (b) Now imagine a sphere of radius 1.3 m that surrounds the box. What is the flux through through this sphere?

2. A long thin wire, hundreds of meters long, carries a uniformly distributed charge of $-8\mu C$ per meter of its length. What is the magnitude and direction of the electric field at a distance of 10.3 m from axis of the wire?

3. A non-conducting sphere of radius R carries a total positive charge q which is distributed uniformly throught its volume. What is the electric field at a distance r from the center when (a) $r < R$ and (b) $r > R$? Plot the magnitude of the electric field as a function of distance when $q = 2\mu C$ and $R = 3m$.

4. A pair of parallel plates carry equal and opposite charge densities and are at a distance d apart. Assume that the charge is distributed uniformly; i.e., the surface charge densities are constant. A particle of mass m and positive charge q is released from rest at the surface of the positively charged plate and strikes the surface of the opposite plate in a time interval t. What is its acceleration? What is the electric field between the plates? What is the magnitude of the electric charge per unit area on the plates? Express your answers in terms of q, m, d, t and constants of nature.

5. Notice the similarity between Coulomb's law and Newton's law of gravitation.(a) State an important difference between the two. (b) What is the analogue of electric field? Find a version of Gauss' Law that applies to gravity. (c) Use it to show that the gravitational field outside a spherical star is the same as if all its mass were concentrated at its center. (d) Would this statement still be true if the star is not spherical?

Solutions

1. The point here is that the dimensions of the box are irrelevant: only the total electric flux is needed to find the electric charge:

$$Q = \epsilon_0 \Phi = 8.9 \times 10^{-12} \times 2000 = 1.8 \times 10^{-8} C.$$

By Gauss' law the flux through the sphere is the same as that through the box: $2000 N m^2 C^{-1}$.

2. This uses Gauss' law also. The electric field has to point in the direction perpendicular to the axis of the wire. Imagine a cylinder of radius r and a large length equal to that of the wire. The electric field points along the normal to the surface at every point. Because all points on the surface are at the same distance from the axis, the electric field will have constant magnitude. So the total flux is the area of the cylinder times the electric field.

$$\Phi = E 2\pi r L$$

The total charge on the wire is $Q = \lambda L$ where λ is the charge per unit legth. By Gauss' law

$$Q = \epsilon_0 \Phi$$

so that

$$\lambda L = \epsilon_0 E 2\pi r L$$

and

$$E = \frac{\lambda}{2\pi \epsilon_0 r}$$

In our numerical example, the charge is negative so the electric field points towards the axis of the wire. It is of magnitude

$$\frac{8 \times 10^{-6}}{2\pi \times 8.9 \times 10^{-12} \times 10.3} = 1.4 \times 10^4 N C^{-1}.$$

3. The electric field has to point along the radius vector as the charge is distrubuted in a rotationally symmetric way. The direction is outwards from the center of the sphere because the charge is positive. The flux through a sphere of radius r is

$$\Phi = 4\pi r^2 E(r)$$

because the field is normal to the sphere at every point and of constant magnitude. By Gauss' law

$$\Phi = \frac{Q(r)}{\epsilon_0}$$

where $Q(r)$ is the charge inside a sphere of radius r. Hence the magnitude of the electric field at a distance r is
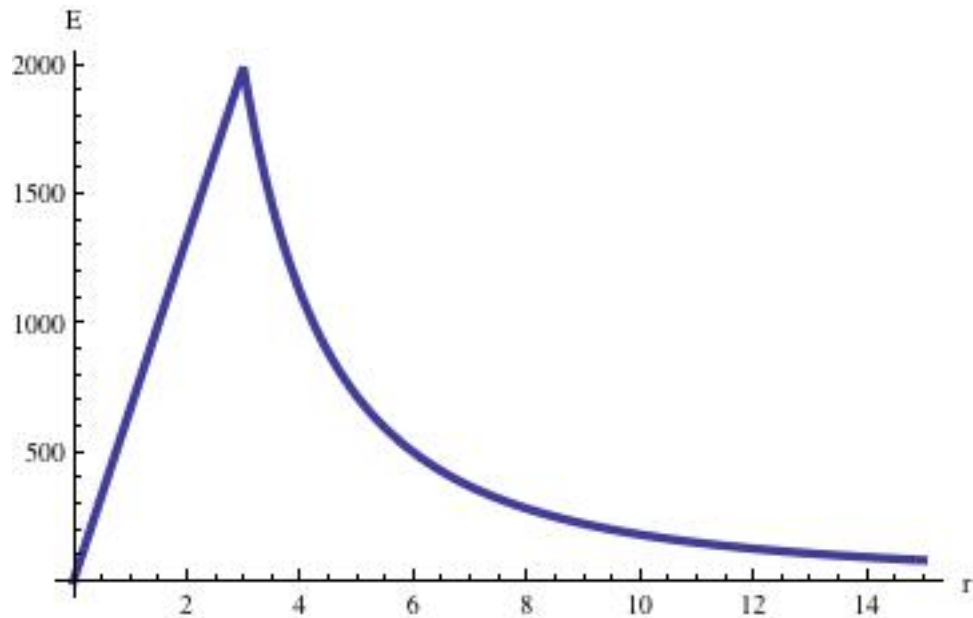
$$E(r) = \frac{Q(r)}{4\pi \epsilon_0 r^2}.$$

Figure 1:

The charge is distributed uniformly; so the density is

$$\rho = \frac{q}{\frac{4\pi}{3}R^3}$$

When $r < R$, the total charge contained inside is

$$Q(r) = \frac{4\pi}{3}r^3\rho = q\frac{r^3}{R^3}.$$

Thus,

$$E(r) = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3}, \quad \text{for } r < R$$

The charge inside a sphere of large radius $r > R$ is just equal to the total charge q. Thus the electric field is

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2} \quad \text{for } r > R$$

as if all the charge were concentrated at the center of the sphere. The required plot is in Figure 1.

4. The electric field in between two oppositely charged parallel plates, each of uniform charge density of magnitude σ is a constant

$$E = \frac{\sigma}{\epsilon_0}.$$

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It points from the positive plate to the negative plate. This follows from Gauss' law. Thus, the force qE and acceleration a of a charged particle in this region are constant. If the particle starts at rest, the distance travelled in time t is

$$d = \frac{1}{2}at^2$$

Thus

$$a = \frac{2d}{t^2}.$$

So the force is

$$F = ma = \frac{2md}{t^2}.$$

The electric field is

$$E = \frac{F}{q} = \frac{2md}{qt^2}.$$

Thus the charge density is

$$\sigma = \epsilon_0 E = \frac{2\epsilon_0 md}{qt^2}.$$

5. Coulomb's law for electrical force states that $F = k\frac{QQ'}{r^2}$ where Q, Q' are charges and r the distance between them; while Newton's law of gravitation states that $F = G\frac{MM'}{r^2}$ where M, M' are masses. The constant of natute k is similar to Newton's constant G.

(a) The direction of gravity is always attractive; for electricity, it is attractive for opposite charges and repulsive for charges of the same sign. This is an important difference.

(b) The electric field at a point is the force per unit charge placed at that point. The analogue for gravity is the force per unit mass at any point. This is the acceleration due to gravity $\mathbf{g}$. A point mass M_1 located at $\mathbf{r}_1$, creates a gravitational field at $\mathbf{r}$ equal to

$$\mathbf{g}(\mathbf{r}) = -G\frac{M_1}{|\mathbf{r} - \mathbf{r}_1|^3}(\mathbf{r} - \mathbf{r}_1)$$

just as for Coulomb's law. There is a gravitational flux $\mathbf{g} \cdot d\mathbf{S}$ through any small surface element $d\mathbf{S}$; recall that $d\mathbf{S}$ is a vector whose magnitude is the area and the direction is normal to the surface. The total gravitational flux through any closed surface is the sum over fluxes through such small elements $\int \mathbf{g} \cdot d\mathbf{S}$. By analogy to the proof of Gauss' law we can see that this is proportional to the total mass contained inside the surface:

$$\int \mathbf{g} \cdot d\mathbf{S} = -4\pi GM$$

(c) If the star is spherical, its mass density is unchanged under rotations around the center of the star. Imagine a sphere with the same center as the

star, with a radius large enough that it is outside the star. Because of the symmetry under rotations, gravitational field has to point radially inward at every point on the sphere. Also, it must have the same magnitude at every point on the sphere. So the total flux is

$$-g(r)4\pi r^2 = -4\pi GM$$

Thus the gravitational field has magnitude

$$g(r) = \frac{GM}{r^2}$$

Its magnitude and direction (radially inward) are as if all the mass of the star where located at the center of the star. Newton knew this fact, but he proved it by a much more complicated method: Gauss found this simpler proof many decades later.

(d) The gravitational field of a non-spherical star is not the same as that of a point mass at its center. The argument above breaks down, because the gravitational field is not constant at all points on the sphere: some points on it are closer to mass than others. For example, the Sun is not exactly spherical: it bulges out at its equator. There is a small correction to planetary orbits because of this. Same for the Earth and satellite orbits.