S11 PHY114 Problem Set 4

S. G. Rajeev

February 21, 2011

Due Monday 21 Feb 2011

1. An isolated capacitor C_1 carries a charge Q_0. It is then connected by conducting wires to a second capacitor C_2 which was previously uncharged. What is the charge on each capacitor now?

2.

(i) How much energy is needed to transfer a small amount of charge dQ from one plate of a capacitor to the other, if the potential difference between the plates is V?

(ii) By thinking of the charge on the capacitor as built up from zero to Qin small increments, what is the energy of a capacitor with charge Q? The capacitance C is constant. Express this energy in terms of $C$and the final voltage V.

3. Recently there have been advances in developing capacitors with very large capacitances of thousands of Farads. They also have a small internal resistance.

(i) How much electrical energy is stored in a 1200 Farad capacitor charged to 12 Volts?

(ii) It is found that the time it takes for half the charge in this capacitor to be discharged is $0.52s$. What is the resistance ?

4. Using the formula for the energy of a parallel plate capacitor, derive a formula for the energy density of the electric field. Assume that the capacitor is made of two parallel conducting plates, each of area A, separated by at a distance d of empty space.

5. A $2.8k\Omega$ and $3.7k\Omega$ resistor are connected in parallel; this combination is then connected in series with a $1.8k\Omega$ resistor. What is the maximum voltage that can be applied across this network if the power dissipated in any one resistor cannot exceed one Watt ?

Solutions

1. Let the charges on the capacitors after they are connected by Q_1 and Q_2 respectively. By conservation of charge

$$Q_1 + Q_2 = Q_0$$

After they are connected by wires the potential difference between plates has to be the same. Thus

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Solving,

$$Q_1 = \frac{C_1 Q_0}{C_1 + C_2}, \quad Q_2 = \frac{C_2 Q_0}{C_1 + C_2}.$$

2. (i) The energy needed is $V dQ = \frac{Q dQ}{C}$.

(ii) The total energy of a capacitor is then $\int_0^Q \frac{Q dQ}{C} = \frac{1}{2C} Q^2$. Since $Q = CV$, this energy is also equal to $\frac{1}{2} CV^2$.

3.

(i)The energy of the capacitor is

$$\frac{1}{2} CV^2 = \frac{1}{2}(1200)(12)^2 = 86kJ$$

(ii) The charge after a time t is

$$Q(t) = Q_0 e^{-\frac{t}{RC}}.$$

Thus when $t = 0.52$s,

$$e^{-\frac{t}{RC}} = \frac{1}{2}$$

$$\frac{t}{RC} = \ln 2$$

$$R = \frac{t}{C \ln 2} = \frac{0.52}{1200 \times \ln 2} = 0.6 m\Omega.$$

4. If the parallel plates carry charges $Q$and $-Q$, the electric field in between the plates is

$$E = \frac{Q}{\epsilon_0 A}$$

uing Gauss' law. The potential difference is

$$V = Ed = \frac{Qd}{\epsilon_0 A}$$

so that the capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

The energy of a capacitor is

$$\frac{1}{2}CV^2 = \frac{1}{2}\left[\frac{\epsilon_0 A}{d}\right]E^2 d^2 = \frac{1}{2}\epsilon_0 E^2 \left[Ad\right].$$

But Ad is the volume of the empty space in between the capacitors. So the energy per unit volume of the electric field in between the two plates is

$$\frac{1}{2}\epsilon_0 E^2.$$

5. The tricky part here is that we must find the power dissipated in each resistor, not just the total. So we must find the current through each resistor.

Let Ibe the total current across the circuit. The power dissipated in the resistor R_3 connected in series is $R_3 I^2$. Let I_1 and I_2 be the currents through the two resistors R_1 and R_2 in parallel. The power dissipated in each resistor is $R_1 I_1^2$ and $R_2 I_2^2$

Then

$$I = I_1 + I_2$$

The voltage across them is equal

$$I_1 R_1 = I_2 R_2$$

Solving

$$I_1 = \frac{R_2}{R_1 + R_2} I, \quad I_2 = \frac{R_1}{R_1 + R_2} I$$

Thus the power dissipated through the three resistors are

$$R_1 I_1^2 = \frac{R_1 R_2^2}{(R_1 + R_2)^2} I^2, \quad R_2 I_2^2 = \frac{R_2 R_1^2}{(R_1 + R_2)^2} I^2, \quad R_3 I^2.$$

All of these have to be less than $1W$.

Putting numbers in $R_1 = 2.8k\Omega, R_2 = 3.7k\Omega, R_3 = 1.8k\Omega$.

$$\frac{R_1 R_2^2}{(R_1 + R_2)^2} = \frac{2.8 \times (3.7)^2}{(2.8 + 3.7)^2} = .9k\Omega$$

$$\frac{R_2 R_1^2}{(R_1+R_2)^2} = \frac{3.7 \times (2.8)^2}{(2.8+3.7)^2} = 0.68k\Omega$$

The strongest condition comes from R_3.

$$1.8 \times 10^3 I^2 < 1$$

$$I < \sqrt{\frac{1}{1800}} = 0.02A.$$

This is the maximum current. To convert this into a voltage we need the resistance of the entire circuit. The two resistors in parallel combine to a resistance of $\frac{R_1 R_2}{R_1+R_2}$ to which we add the resistor in series

$$R = R_3 + \frac{R_1 R_2}{R_1 + R_2} = 1.8 + \frac{2.8 \times 3.7}{2.8 + 3.7} = 3.4k\Omega$$

Thus the voltage is

$$RI < 68V.$$