Small Oscillations of the n-Pendulum and the "Hanging Rope" Limit $n \to \infty$

Ryan Rubenzahl
University of Rochester
Professor S. G. Rajeev
January 2, 2017

Abstract

In this paper we will extend the solutions of the single, double, and triple pendulum to a system of arbitrary n pendulums each hanging below the previous, and explore the equations of motion for small oscillations about the equilibrium position. We will approach this problem both from a Newtonian Mechanics and Lagrangian Dynamics perspective. We will also explore the resulting equations of motion for small oscillations of a hanging rope of constant mass density by both taking the limit $n \to \infty$ in our solution for the n-pendulum, and by formulating Lagrangian Dynamics for a continuous system. We will show that all of these approaches result in identical equations of motion and are therefore equally valid ways of approaching this problem. Finally, we will present a numerical solution to the nonlinear equations of motion for the n-pendulum and demonstrate chaotic behavior through *Mathematica* animations.

Contents

1 Introduction **2**

2 Newtonian Approach **3**
2.1 Equations of Motion for Finite n . . . 3
2.1.1 Equations of Motion in the Limit $n \to \infty$ . . . 5
2.2 Equations of Motion for a Continuous Mass-Density Rope . . . 6

3 Lagrangian Approach **9**
3.1 Equations of Motion for Finite n . . . 9
3.1.1 The Lagrangian and Equations of Motion . . . 9
3.1.2 Frequencies of Small Oscillation and the Normal Modes . . . 14
3.2 Equations of Motion for a Continuous Mass-Density Rope . . . 16
3.2.1 The Lagrangian Formulation for a Continuous System . . . 16
3.2.2 Equations of Motion Using the Lagrangian Density . . . 17

4 Numerical Solution to Nonlinear Equations of Motion for Finite n **18**

5 Conclusion **20**

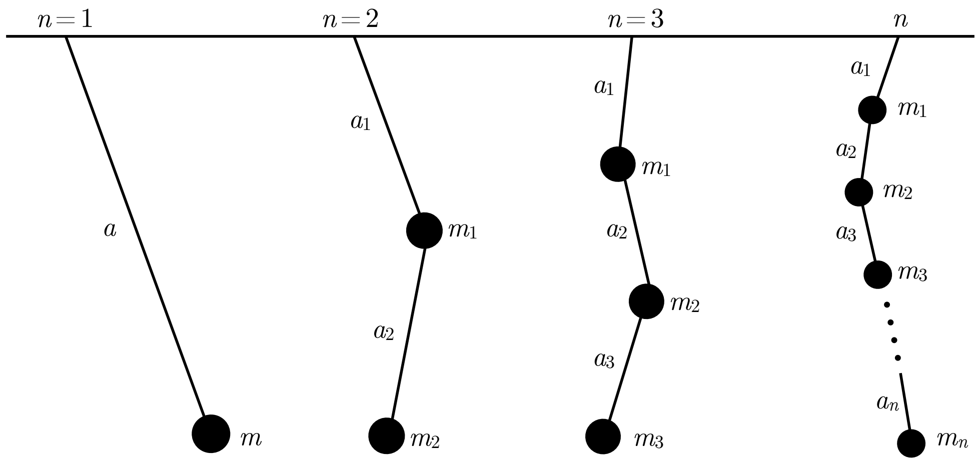

Figure 1.1: Sketch of the n-pendulum system for $n = 1, n = 2, n = 3$, and arbitrary n.

1 Introduction

The simple pendulum, consisting of a mass-bob suspended by a rigid rod allowed to pivot about the suspension point, is one of the most iconic systems in physics. Its properties have been studied extensively for hundreds of years and is an iconic demonstration included in every introductory physics course. Many variations on the simple pendulum have been studied over the years that have practical applications useful for timekeeping or engineering purposes, such as Huygen's pendulum and Foucault's pendulum. Other variations pose physically interesting problems to explore that test our understanding of the physical world, like the Kapitza pendulum.

An example of such a "physically interesting" variation of the simple pendulum involves hanging another simple pendulum from the first. This system, dubbed the "double pendulum," was explored in class. As a homework assignment we solved the "triple pendulum," which includes a third pendulum hung from the second. In this paper, we will extend the analysis of these two systems to an arbitrary number of n many pendulums each suspended from the mass-bob of the pendulum above it. As the sketch of these systems in Figure 1.1 shows, we will index the mass and arm length of the i^{th} bob as m_i and a_i respectively, where $i = 0$ refers to the fixed point at the top, $i = 1$ refers to the top-most pendulum, and $i = n$ describes the bottom-most pendulum.

Throughout our analysis we will carry out as much of the derivation as we easily can with as few assumptions as possible. In our final solutions however, we will use the following simplifying assumptions:

1. The mass of each pendulum bob is equal. That is, $m_i = m \ \forall \ i$.
2. The arm length of each pendulum is equal. That is, $a_i = a \ \forall \ i$.
3. In the limit $n \to \infty$, we keep the total length of the system fixed. That is, $\ell = na$ is constant.
4. In the "small oscillations" linear approximation, described by $\theta_i \ll 1 \ \forall \ i$, we will keep terms in the Taylor expansion up to second order.

We will explore two methods of solving this problem that will result in equivalent solutions, each of which will be approached from a Newtonian and Lagrangian Mechanics perspective:

1. Solving for the equations of motion for finite n (discrete case), then taking the limit $n \to \infty$.
2. First taking the limit $n \to \infty$ (continuous case), then solving for the equations of motion.

Finally, we will observe the chaotic behavior of the full nonlinear solution by solving the equations of motion numerically and producing an animation of the motion using *Mathematica*.

2 Newtonian Approach

The approach using Newtonian Mechanics is centered around determining all of of the forces acting on the given system and using Newton's second law

$$\vec{F}_{net} = \sum_i \vec{F}_i = m\vec{a} \qquad \text{(Newton's 2nd Law)}$$

where $\vec{F}_{net}$ is the sum of all of the forces $\vec{F}_i$ acting on one particle in the system, m is the mass of the particle, and $\vec{a}$ is the particle's acceleration. The acceleration may also be denoted as

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}}$$

where $\vec{x}$ is the position vector of the particle and each dot denotes a time derivative. We will use the dot notation to represent differentiation with respect to time throughout the rest of this paper.

2.1 Equations of Motion for Finite n

We first need to determine an adequate coordinate system to describe the positions of each mass-bob. Using the indexing convention shown in Figure 1.1, we call θ_i the angle between the vertical and the i^{th} arm. Define the origin $(x_0, y_0) = (0, 0)$ to be at the fixed pivot point at $i = 0$. Let the line along which the pendulum would lie along if it were at rest define the y-axis, and set the x-axis perpendicular to this plane. Then we can also define the coordinates (x_i, y_i) such that x_i is the horizontal displacement from the y-axis and y_i is the y-coordinate of the i^{th} mass-bob. These coordinates are displayed in Figure 2.1.

Each mass-bob will experience a torque due to the gravitational force if displaced from the equilibrium position. The magnitude of the downward force pulling on the i^{th} mass-bob will be due to the weight of all of the other masses below it, which we will denote as the tension T_i by

$$T_i = \sum_{j=i+1}^{n} m_j g.$$

The horizontal component of T_i will be the harmonic restoring force $\vec{F}_i$ pointing towards the y-axis that brings the pendulum back to its equilibrium position, with magnitude:

$$F_i = T_i \tan\theta_i = \sum_{j=i+1}^{n} m_j g \tan\theta_i. \tag{2.1}$$

However, the mass-bob above will also be subject to its own restoring force due to all of the weight below it, and so it too will be moving. In other words, the pivot point of the i^{th} mass-bob is the $(i-1)^{th}$ bob, which is in motion itself, with the lone exception of the "zeroth" bob (the origin). Thus the difference in the magnitudes of the forces acting on the i^{th} and $(i-1)^{th}$ bob will create the resulting net-force by Newton's second law that will result in the motion of the i^{th} pendulum. For small oscillations, we can approximate the motion as purely in the x direction. Thus Newton's second law becomes

$$F_{net} = F_i - F_{i-1} = \sum_{j=i+1}^{n} m_j g \tan\theta_i - \sum_{j=i}^{n} m_j g \tan\theta_{i-1} = m\ddot{x}_i. \tag{2.2}$$

We can simplify this further by making use of Assumptions 1 and 4 described in Section 1, yielding

$$(n-i)mg\theta_i - (n-i+1)mg\theta_{i-1} = m\ddot{x}_i, \tag{2.3}$$

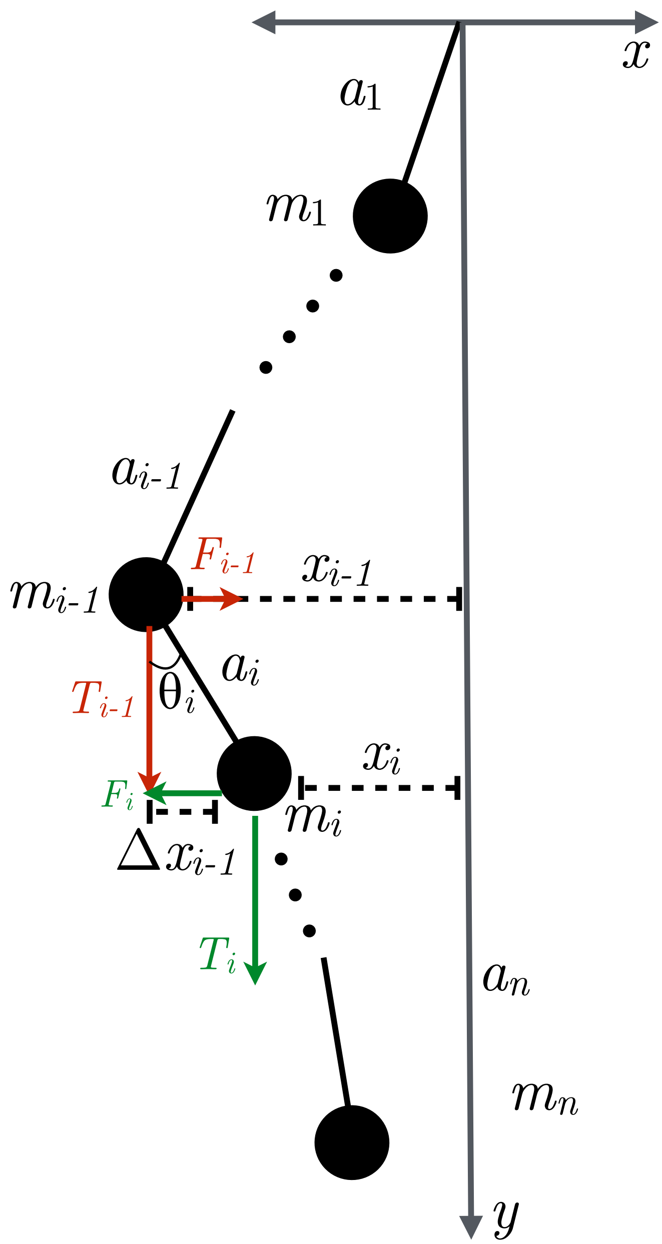

Figure 2.1: Forces acting on the i^{th} and $(i-1)^{th}$ mass-bob caused by the tension due to all masses below that point. The difference in the forces between neighbors will produce the net force, causing the motion of that particular mass-bob. The combined interactions of all the mass-bobs result in the overall motion of the system.

where we have used the Taylor series approximation $\tan\theta \approx \theta$ for $\theta \ll 1$. Now, from Figure 2.1, it can be seen that for small angles the angle θ_i can be approximated as

$$\theta_i = \frac{\Delta x_i}{a_{i+1}}$$

where

$$\Delta x_i = x_{i+1} - x_i$$

is the difference in x coordinates between the mass bob and the one below it. It is this angle that defines the horizontal component of the tension. Making these substitutions into equation (2.2) we find

$$(n-i)mg\left(\frac{x_{i+1}-x_i}{a_{i+1}}\right) - (n-i+1)mg\left(\frac{x_i - x_{i-1}}{a_i}\right) = m\ddot{x}_i. \tag{2.4}$$

We can cancel the mass m from both sides and rearrange terms on the left side to yield

$$\boxed{\ddot{x}_i = g\left[(n-i)\left(\frac{x_{i+1}-x_i}{a_{i+1}} - \frac{x_i - x_{i-1}}{a_i}\right) - \left(\frac{x_i - x_{i-1}}{a_i}\right)\right]} \tag{2.5}$$

which describes the motion of the i^{th} pendulum. This equation was first found by Daniel Bernoulli and then Euler in the 1730s, and later by Johann Bernoulli in 1742, who all worked independently to solve the problem of the then called "hanging chain" [1]. As we will see in the next section, when we take the limit to the continuous case, we will find what Bernoulli first discovered when he solved this problem.

2.1.1 Equations of Motion in the Limit $n \to \infty$

First, we rewrite equation (2.5) as the following by multiplying the first term in the square brackets by $\frac{a}{a}$ and applying Assumption 2:

$$\ddot{x}_i = g\left[(n-i)\frac{a}{a}\left(\frac{x_{i+1}-x_i}{a_{i+1}} - \frac{x_i - x_{i-1}}{a_i}\right) - \left(\frac{x_i - x_{i-1}}{a_i}\right)\right]. \tag{2.6}$$

As we take the limit of equation (2.5) for $n \to \infty$, the coordinates of each mass-bob will become the function describing the shape of the now rope of constant mass density with time. That is,

$$\lim_{n\to\infty} x_i = x = x(y,t).$$

Thus

$$\lim_{n\to\infty} \ddot{x}_i = \frac{\partial^2 x(y,t)}{\partial t^2}.$$

By Assumption 3 we will also have

$$\lim_{n\to\infty} (n-i)a = (\ell - y).$$

By the definition of the derivative, the remaining terms become

$$\lim_{n\to\infty}\left(\frac{x_i - x_{i-1}}{a_i}\right) = \frac{\partial x(y,t)}{\partial y},$$

$$\lim_{n\to\infty}\frac{1}{a}\left(\frac{x_{i+1}-x_i}{a_{i+1}} - \frac{x_i - x_{i-1}}{a_i}\right) = \frac{\partial^2 x(y,t)}{\partial y^2}.$$

Thus equation (2.6) becomes

$$\boxed{\frac{\partial^2 x}{\partial t^2} = g\left[(\ell - y)\frac{\partial^2 x}{\partial y^2} - \frac{\partial x}{\partial y}\right]} \tag{2.7}$$

We have recovered the wave equation, albeit in a different form than we are used to! We will now derive this exact equation starting from the continuous system, and solve it for $x(y,t)$.

2.2 Equations of Motion for a Continuous Mass-Density Rope

This will be a very similar treatment of the problem as in Section 2.1, but nevertheless we will demonstrate the "correctness" of the result of taking the limit by solving the problem from a continuous setup. Consider a vertically hanging rope of constant mass density μ that is hanging from one end with the other free to move. Let $x = x(y,t)$ describe the displacement from the vertical axis at a point y along the vertical axis at time t. The tension at any point in the rope is due to all the weight of the rope hanging below that point. Focusing our attention on an infinitesimal piece of the rope of length ds at a distance y along the rope, this piece is subject to the tension force

$$T(y) = \mu(\ell - y)g$$

in the case of small oscillations, so that to first order the arclength s and coordinate y are equal. Again, the restoring force will be the horizontal component of the tension:

$$F = T(y)\tan\theta = T(y)\frac{\partial x}{\partial y}. \tag{2.8}$$

For an infinitesimal chunk of rope, Newton's second law reads

$$dF = dm\ddot{x} = \mu dy\ddot{x}$$

or equivalently

$$\mu\ddot{x} = \frac{dF}{dy}.$$

Combining with equation (2.8) we have

$$\mu\ddot{x} = \mu\frac{\partial^2 x}{\partial t^2} = \frac{d}{dy}\left(T(y)\frac{\partial x}{\partial y}\right) = \frac{d}{dy}\left(\mu(\ell - y)g\frac{\partial x}{\partial y}\right) = -\mu g\frac{\partial x}{\partial y} + \mu g(\ell - y)\frac{\partial^2 x}{\partial y^2},$$

and by cancelling μ from both sides we recover

$$\frac{\partial^2 x}{\partial t^2} = g\left[(\ell - y)\frac{\partial^2 x}{\partial y^2} - \frac{\partial x}{\partial y}\right] \tag{2.9}$$

which is exactly equation (2.7) as expected. To solve this differential equation, we will use the method of separation of variables and make the separable ansatz

$$x(y,t) = \phi(y)G(t).$$

Substituting this into (2.9) and dividing both sides by $\phi(y)G(t)$ yields

$$\frac{1}{G}\frac{dG^2}{\partial t^2} = g\left[(\ell - y)\frac{\partial^2\phi}{\partial y^2} - \frac{\partial\phi}{\partial y}\right]\frac{1}{\phi} = -\omega^2$$

where we choose the separation constant $-\omega^2$ in anticipation of a harmonic oscillator. This gives us the following two ordinary differential equations:

$$\frac{d^2G}{dt^2} + \omega^2 G = 0 \tag{2.10}$$

$$(\ell - y)\frac{d^2\phi}{dy^2} - \frac{d\phi}{dy} + \frac{\omega^2}{g}\phi = 0 \tag{2.11}$$

Equation (2.10) is solved by a harmonic oscillator

$$G(t) = A\cos(\omega t - \phi) \tag{2.12}$$

so we can see that ω has the meaning of frequency. For equation (2.11), we make the change of variables

$$z^2 = \frac{4}{g}(\ell - y).$$

The derivatives transform as

$$\begin{aligned}
\frac{d\phi}{dy} &= \frac{d\phi}{dz}\frac{dz}{dy} = \frac{-2}{gz}\frac{d\phi}{dz}, \\
\frac{d^2\phi}{dy^2} &= \frac{d}{dy}\left(\frac{-2}{gz}\frac{d\phi}{dz}\right) = \frac{4}{g^2z^2}\frac{d^2\phi}{dz^2} - \frac{4}{g^2z^3}\frac{d\phi}{dz}.
\end{aligned}$$

Substituting these into (2.11) gives

$$\frac{gz^2}{4}\left(\frac{4}{g^2z^2}\frac{d^2\phi}{dz^2} - \frac{4}{g^2z^3}\frac{d\phi}{dz}\right) + \frac{2}{gz}\frac{d\phi}{dz} + \frac{\omega^2}{g}\phi = 0.$$

Distributing through the first term gives

$$\frac{1}{g}\frac{d^2\phi}{dz^2} - \frac{1}{gz}\frac{d\phi}{dz} + \frac{2}{gz}\frac{d\phi}{dz} + \frac{\omega^2}{g}\phi = 0,$$

and multiplying everything by gz^2 we get

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + \omega^2z^2\phi = 0, \tag{2.13}$$

which is Bessel's equation of order zero, first encountered in this context by Daniel Bernoulli in 1733 [1]! The solution to (2.13) is a linear combination of the zeroth order Bessel functions

$$\phi(z) = AJ_0(\omega z) + BY_0(\omega z).$$

Our solution must remain finite at the end of the rope corresponding to $y = \ell$ or $z = 0$, thus since $Y_0(\omega z)$ is singular at the origin the coefficient B must vanish. So our solution in y is

$$\phi(y) = AJ_0\left(2\omega\sqrt{\frac{\ell - y}{g}}\right).$$

We can determine the constant ω using the boundary condition at the fixed point

$$\phi(0) = 0.$$

Thus the frequency ω is determined by the equation

$$J_0\left(2\omega\sqrt{\ell/g}\right) = 0, \tag{2.14}$$

which has infinitely many solutions (Bernoulli understood this from the fact that for n linked pendulums, there will be n modes, and in this case we have taken the limit $n \to \infty$ [1]). If we denote z_{0n} as the n^{th} positive solution of

$$J_0(z) = 0$$

then we can express the frequencies as

$$\omega_n = \frac{1}{2}\sqrt{\frac{g}{\ell}}z_{0n}.$$

When Bernoulli derived this expression, he had discovered the problem of finding the zeroes of the zeroth order Bessel function J_0. Both Bernoulli and Euler used their solutions in the discrete case, like we had derived in Section 2.1, to numerically approximate the zeroes by using their solutions as an expansion of the continuous solution. They even go as far as to confirm the first several zeroes experimentally by, of course, measuring the frequencies of oscillation of a hanging chain [1]. Nowadays, however, we have computers at our disposal to crank out the calculations.

Putting the two solutions together, the normal modes of the rope can be expressed as

$$x_n(y,t) = AJ_0\left(2\omega_n\sqrt{\frac{\ell - y}{g}}\right)\cos(\omega_n t - \phi). \tag{2.15}$$

Thus we can write the final solution for the motion of the rope as

$$\boxed{x(y,t) = \sum_{n=1}^{\infty} AJ_0\left(2\omega_n\sqrt{\frac{\ell - y}{g}}\right)\cos(\omega_n t - \phi), \qquad \omega_n = \frac{1}{2}\sqrt{\frac{g}{\ell}}z_{0n}} \tag{2.16}$$

We can make some plots in *Mathematica* to see how the first few normal modes evolve with time. The first six are shown in Figure 2.2.

To briefly summarize, we have used Newtonian mechanics to determine the equations of motion for an arbitrary but finite number of n pendulums, each hanging from the previous (equation (2.5)). We took the limit of this equation to obtain a differential equation describing the motion (equation (2.7)), as well as deriving the differential equation directly approaching the system as a continuous system (equation (2.9)). We solved this differential equation and obtained Bessel functions to describe the mode shapes and a harmonic oscillator to describe the time evolution of the system (equation (2.15)). The frequencies of small oscillations are determined by the zeroes of the zeroth order Bessel function (equation (2.16)). We will now approach the problem using a Lagrangian Mechanics perspective, and compare our solutions to what we have found so far.

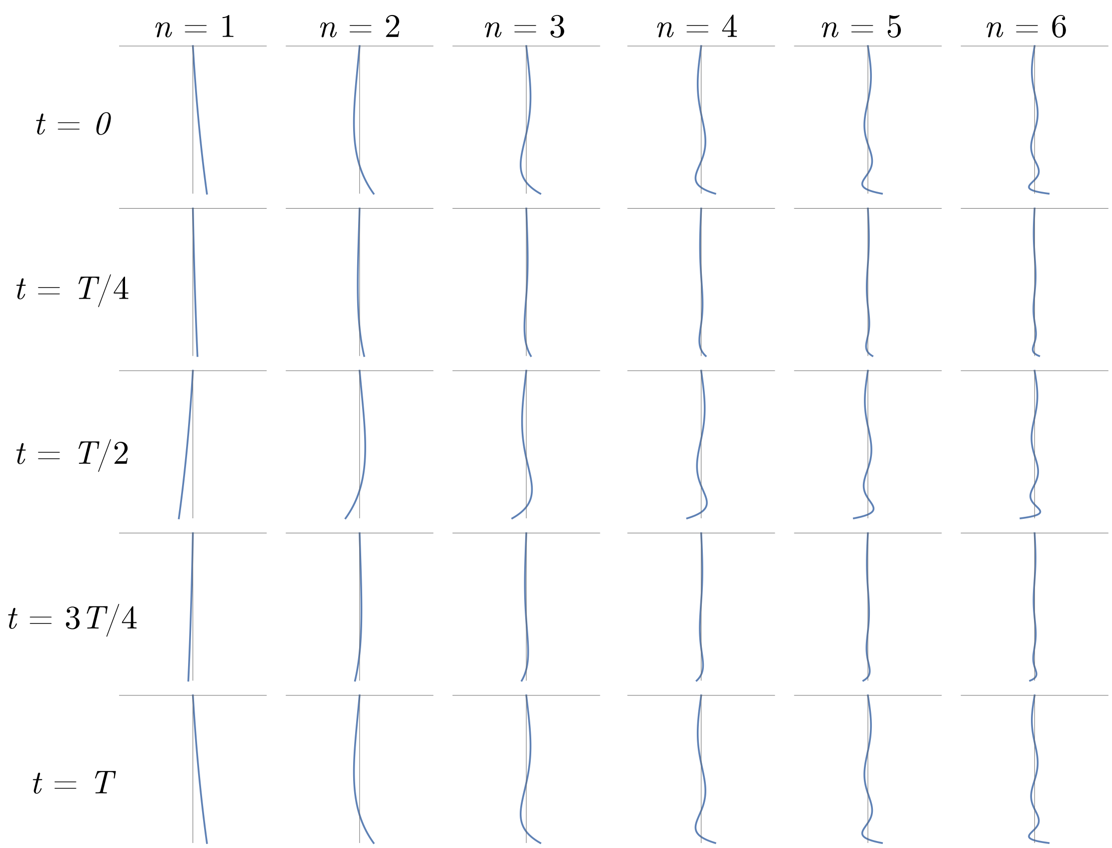

Figure 2.2: The first six normal modes of small oscillations of a hanging rope, given by equation (2.15) at evenly spaced time intervals over the course of one period T of oscillation.

3 Lagrangian Approach

In stark contrast to Newtonian Mechanics, formulating a problem in Lagrangian Mechanics requires no analysis of forces[1]. Instead, the energy of the system and principle of extremal action are used to determine the equations of motion. The principle of extremal actions states that the action, defined as

$$S \equiv \int_{t_1}^{t_2} L(q^i, \dot{q}^i; t)dt$$

of a particle moving between two points is extremized. The function $L(q^i, \dot{q}^i; t)$ of the generalized coordinates q^i and velocities $\dot{q}^i$ is called the Lagrangian of the system and is defined as the difference of the kinetic and potential energies

$$L = T - U.$$

For the action to be minimized, the Lagrangian must satisfy the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

which produce the equations of motion. For the discrete case, where n is finite, we will use this approach in determining the equations of motion. However when we consider the continuous case for the rope, we will need to reformulate the principle of extremal action to apply it to a continuous system.

3.1 Equations of Motion for Finite n

We will use the same coordinates represented in Figure 2.1 to describe the position of each mass-bob. Specifically, we will use the angles θ_i defining the angular displacement from the vertical for each mass-bob as the generalized coordinates for the system. Once we have found the equations of motion, we will use the relation

$$\theta_i = \frac{x_{i+1} - x_i}{a_{i+1}} \tag{3.1}$$

to change coordinates back to the form used in Section 2 so that we may compare to our solutions there.

3.1.1 The Lagrangian and Equations of Motion

We will first derive the explicit Lagrangian for the full nonlinear system, then apply Assumption 4 in the linear approximation. We start by by expressing the cartesian coordinate position of each mass-bob in terms of the angular displacement coordinate θ_i

$$\begin{aligned}
x_1 &= a_1 \sin\theta_1 & y_1 &= a_1 \cos\theta_1 \\
x_2 &= x_1 + a_2 \sin\theta_2 & y_2 &= y_1 + a_2 \cos\theta_2 \\
x_3 &= x_2 + a_3 \sin\theta_3 & y_3 &= y_2 + a_3 \cos\theta_3 \\
&\vdots & &\vdots \\
x_i &= \sum_{j=1}^{i} a_j \sin\theta_j & y_i &= \sum_{j=1}^{i} a_j \cos\theta_j
\end{aligned} \tag{3.2}$$

[1]So long as there are no external forces, in which case the Lagrange multiplier technique and constraint equations are needed.

The velocities in each direction for the i^{th} mass-bob are then

$$\dot{x}_i = \sum_{j=1}^{i} a_j \dot{\theta}_j \cos\theta_j \qquad\qquad \dot{y}_i = -\sum_{j=1}^{i} a_j \dot{\theta}_j \sin\theta_j \tag{3.3}$$

So the overall velocity squared is the sum of the square of equations (3.3a) and (3.3b)

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 = \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \cos\theta_j\right)^2 + \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \sin\theta_j\right)^2.$$

Thus the kinetic energy of the whole system is

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} a_j \dot{\theta}_j \cos\theta_j\right)^2 + \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \sin\theta_j\right)^2\right], \tag{3.4}$$

and the potential energy is, taking $y = 0$ to be the zero of potential energy,

$$U = -g\sum_{i=1}^{n} m_i y_i = -g\sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \cos\theta_j. \tag{3.5}$$

Therefore the Lagrangian for the n-pendulum is

$$\boxed{L = \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} a_j \dot{\theta}_j \cos\theta_j\right)^2 + \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \sin\theta_j\right)^2\right] + g\sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \cos\theta_j} \tag{3.6}$$

We will use this Lagrangian in Section 4 to numerically solve for the full nonlinear equations of motion. Here, however, we will Assumption 4 so that

$$\cos\theta_i \approx 1 - \frac{\theta_i^2}{2} \qquad\qquad \sin\theta_i \approx \theta_i \tag{3.7}$$

Then the Lagrangian in (3.6) will become

$$L \approx \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} a_j \dot{\theta}_j (1 - \frac{\theta_j^2}{2})\right)^2 + \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \theta_j\right)^2\right] + g\sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j (1 - \frac{\theta_j^2}{2}) \tag{3.8}$$

We can drop the constant term from the potential energy term in (3.8) and write the Lagrangian equivalently as

$$L \approx \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} a_j \dot{\theta}_j (1 - \frac{\theta_j^2}{2})\right)^2 + \left(\sum_{j=1}^{i} a_j \dot{\theta}_j \theta_j\right)^2\right] - \frac{1}{2} g\sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \theta_j^2 \tag{3.9}$$

Keeping with Assumption 4, we drop the higher order terms from (3.9) in favor of

$$\boxed{L \approx \sum_{i=1}^{n} \frac{1}{2} m_i \left[\left(\sum_{j=1}^{i} a_j \dot{\theta}_j\right)^2 - g\sum_{j=1}^{i} a_j \theta_j^2\right]} \tag{3.10}$$

where we have also rearranged the summations and some terms to make the expression cleaner.

We can now compute the k^{th} Euler-Lagrange equation for the Lagrangian in (3.10) by first finding each of the partial derivatives. We'll start with

$$\frac{\partial L}{\partial \theta_k} = -\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{2}m_i g a_j \frac{\partial(\theta_j^2)}{\partial \theta_k} = -\sum_{i=1}^{n}\sum_{j=1}^{i} m_i g a_k \theta_k \delta_{jk}$$

where δ_{jk} is the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{(Kronecker delta)}$$

which has the effect of relabeling the summations, yielding

$$\frac{\partial L}{\partial \theta_k} = -a_k \theta_k g \sum_{i=k}^{n} m_i. \qquad (3.11)$$

From Assumptions 1 and 2, we can write (3.11) as

$$\frac{\partial L}{\partial \theta_k} = -(n-k+1) mga\theta_k. \qquad (3.12)$$

Next we find the partial derivative with respect to $\dot{\theta}_k$:

$$\frac{\partial L}{\partial \dot{\theta}_k} = \sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} a_j \dot{\theta}_j\right) \sum_{l=1}^{i} a_l \delta_{lk}.$$

Using Assumptions 1 and 2, we can simplify this as

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_k} &= ma^2 \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \dot{\theta}_j\right)\left(\sum_{l=1}^{i} \delta_{lk}\right) \\
&= ma^2 \sum_{\substack{i=1 \\ i\geq k}}^{n} \sum_{\substack{j=1 \\ j \leq i}}^{i} \dot{\theta}_j && \text{(Relabelling the outer sum)} \\
&= ma^2 \sum_{\substack{j=1 \\ i\geq j}}^{n} \sum_{\substack{i=1 \\ i \geq k}}^{n} \dot{\theta}_j && \text{(Swapping summations)}
\end{aligned}$$

Now, i is an upper bound here for both j and k, so we can combine the inequalities and write this last expression as

$$\frac{\partial L}{\partial \dot{\theta}_k} = ma^2 \sum_{j=1}^{n} \sum_{i=\max(k,j)}^{n} \dot{\theta}_j = ma^2 \sum_{j=1}^{n} (n - \max(k,j) + 1)\dot{\theta}_j.$$

Thus we find

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_k} = ma^2 \sum_{j=1}^{n} (n - \max(k,j) + 1)\ddot{\theta}_j. \qquad (3.13)$$

Together, (3.12) and (3.13) give the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_k} - \frac{\partial L}{\partial \theta_k} = ma^2 \sum_{j=1}^{n} (n - \max(k,j) + 1)\ddot{\theta}_j + (n-k+1)mga\theta_k = 0. \qquad (3.14)$$

Dividing (3.14) by ma^2 we get

$$\boxed{\sum_{j=1}^{n}(n-\max(k,j)+1)\ddot{\theta}_j + (n-k+1)\frac{g}{a}\theta_k = 0} \tag{3.15}$$

which for $n = 1$ recovers the equation of motion for a simple pendulum of length a

$$\ddot{\theta}_1 + \frac{g}{a}\theta_1 = 0 \qquad \text{(Simple pendulum)}$$

Equation (3.15) gives the equations of motion for small oscillations of the n-pendulum. To compare to the equations of motion found in Section 2.1 (equation (2.5)), we use the relation given in equation (3.1) and rewrite (3.15) as

$$\sum_{i=1}^{n}(n-\max(k,i)+1)(\ddot{x}_i - \ddot{x}_{i-1}) + g(n-k+1)\frac{x_i - x_{i-1}}{a} = 0 \tag{3.16}$$

where we have also relabelled the summation index j to i and multiplied both sides by a factor of a. Each term in the summation here is subtracting term before it, so to see how things may cancel we can write out the summation into three parts:

$$\underbrace{\sum_{i=1}^{k-1}(n-k+1)(\ddot{x}_i - \ddot{x}_{i-1})}_{i<k \text{ terms}} + \underbrace{(n-k+1)(\ddot{x}_k - \ddot{x}_{k-1})}_{i=k \text{ term}} + \underbrace{\sum_{i=k+1}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1})}_{i>k \text{ terms}} + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0 \tag{3.17}$$

We'll first look at how the first group of terms ($i < k$) behave by writing out the last several terms explicitly:

$$\sum_{i=1}^{k-1}(n-k+1)(\ddot{x}_i - \ddot{x}_{i-1}) = (n-k+1)\left(\sum_{i=1}^{k-4}(\ddot{x}_i - \ddot{x}_{i-1}) + (\cancel{\ddot{x}_{k-3}} - \ddot{x}_{k-4}) + (\cancel{\ddot{x}_{k-2}} - \cancel{\ddot{x}_{k-3}}) + (\ddot{x}_{k-1} - \cancel{\ddot{x}_{k-2}})\right)$$

Writing the sum out like this allows us to see that the sum is telescoping as the $\ddot{x}_{i-1}$ term cancels the $\ddot{x}_i$ term in the piece before it, leaving behind only the first term of the last piece $\ddot{x}_{k-1}$ and the last term of the first piece $\ddot{x}_0$. However, the latter term is zero since $i = 0$ corresponds to the the fixed pivot point at the top of the system. Thus the entire summation reduces down to just

$$\sum_{i=1}^{k-1}(n-k+1)(\ddot{x}_i - \ddot{x}_{i-1}) = (n-k+1)\ddot{x}_{k-1}. \tag{3.18}$$

If we substitute this into (3.17), we can see that the second term in the $i = k$ term cancels (3.18), so (3.17) becomes

$$(n-k+1)\ddot{x}_k + \sum_{i=k+1}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0. \tag{3.19}$$

To reduce this, we use a similar treatment for the $i > k$ summation by writing out the first two terms as follows:

$$\begin{aligned}\sum_{i=k+1}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) &= (n-k)(\ddot{x}_{k+1} - \ddot{x}_k) + (n-k-1)(\ddot{x}_{k+2} - \ddot{x}_{k+1}) + \sum_{i=k+3}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) \\ &= (n-k)(\cancel{\ddot{x}_{k+1}} - \ddot{x}_k) + (n-k)(\ddot{x}_{k+2} - \cancel{\ddot{x}_{k+1}}) - (\ddot{x}_{k+2} - \ddot{x}_{k+1}) + \sum_{i=k+3}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) \\ &= -(n-k)\ddot{x}_k + (n-k)\ddot{x}_{k+2} - \ddot{x}_{k+2} + \ddot{x}_{k+1} + \sum_{i=k+3}^{n}(n-i+1)(\ddot{x}_i - \ddot{x}_{i-1})\end{aligned}$$

We'll write out one more term, so the last line becomes

$$
\begin{aligned}
&= -(n-k)\ddot{x}_k + (n-k)\ddot{x}_{k+2} - \ddot{x}_{k+2} + \ddot{x}_{k+1} + (n-k-2)(\ddot{x}_{k+3} - \ddot{x}_{k+2}) + \sum_{i=k+4}^{n} (n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) \\
&= -(n-k)\ddot{x}_k + \cancel{(n-k)\ddot{x}_{k+2}} - \ddot{x}_{k+2} + \ddot{x}_{k+1} + (n-k)(\ddot{x}_{k+3} - \cancel{\ddot{x}_{k+2}}) - 2(\ddot{x}_{k+3} - \ddot{x}_{k+2}) \\
&\qquad + \sum_{i=k+4}^{n} (n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) \\
&= -(n-k)\ddot{x}_k + \ddot{x}_{k+1} + \ddot{x}_{k+2} + (n-k)\ddot{x}_{k+3} - 2\ddot{x}_{k+3} + \sum_{i=k+4}^{n} (n-i+1)(\ddot{x}_i - \ddot{x}_{i-1})
\end{aligned}
$$

So each piece of the summation has its lower term cancel the higher term from the piece below it, but always while leaving behind one factor of $\ddot{x}_i$, since we can always factor out the $(n-k)$ part to make pieces cancel. Since the sum runs to n, the final piece will have a coefficient of $(n-n+1) = 1$, so there will be one final copy of $\ddot{x}_n$ left at the end as expected by the pattern. Thus the sum can be rewritten as

$$\sum_{i=k+1}^{n} (n-i+1)(\ddot{x}_i - \ddot{x}_{i-1}) = -(n-k)\ddot{x}_k + \sum_{i=k+1}^{n} \ddot{x}_i. \tag{3.20}$$

Then substituting this back into (3.19), and also rewriting the first term of (3.19) as $(n-k+1)\ddot{x}_k = \ddot{x}_k + (n-k)\ddot{x}_k$ we have

$$\ddot{x}_k + (n-k)\ddot{x}_k - (n-k)\ddot{x}_k + \sum_{i=k+1}^{n} \ddot{x}_i + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0, \tag{3.21}$$

or

$$\ddot{x}_k + \sum_{i=k+1}^{n} \ddot{x}_i + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0. \tag{3.22}$$

Notice that (3.22) can be rewritten as

$$
\begin{aligned}
&\sum_{i=k}^{n} \ddot{x}_i + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0 \\
\rightarrow &\sum_{i=k}^{n} \ddot{x}_i = -g(n-k+1)\frac{x_k - x_{k-1}}{a}.
\end{aligned} \tag{3.23}
$$

and substituting (3.23) into (3.22) and keeping track of the indices we have

$$\ddot{x}_k - g(n-k)\frac{x_{k+1} - x_k}{a} + g(n-k+1)\frac{x_k - x_{k-1}}{a} = 0, \tag{3.24}$$

which can easily be rearranged to find

$$\boxed{\ddot{x}_k = g\left[(n-k)\left(\frac{x_{k+1} - x_k}{a} - \frac{x_k - x_{k-1}}{a}\right) - \frac{x_k - x_{k-1}}{a}\right]} \tag{3.25}$$

which describes the motion of the k^{th} pendulum and is exactly equation (2.5) with index k rather than i. Notice here the a's do not have indices as we made use of Assumption 2 much earlier here than we did in Section 2.1. Thus we have arrived at the same conclusion using Lagrangian Mechanics as we did with Newtonian Mechanics, albeit with quite a bit more work[2].

[2]This is not usually the case!

3.1.2 Frequencies of Small Oscillation and the Normal Modes

For a system with many degrees of freedom, such as the n-pendulum, the natural frequencies of oscillation ω are given by the eigenvalue problem with characteristic equation

$$\det(K - \omega^2 M) = 0, \tag{3.26}$$

where K is the Hessian of potential energy with components

$$K_{\mu\nu} = \frac{\partial^2 U}{\partial q^\mu \partial q^\nu}, \tag{3.27}$$

and M is the inertia matrix with components

$$M_{\mu\nu} = \frac{\partial^2 T}{\partial \dot{q}^\mu \partial \dot{q}^\nu}. \tag{3.28}$$

Returning to the Lagrangian from equation (3.10), the kinetic energy and potential energy of the n-pendulum for small oscillations are given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} a_j \dot{\theta}_j\right)^2 \tag{3.29}$$

$$U = \frac{1}{2} g \sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \theta_j^2 \tag{3.30}$$

From this we can determine the matrices K and M:

$$K_{\mu\nu} = \frac{\partial^2 U}{\partial \theta_\mu \partial \theta_\nu} = \frac{1}{2} g \sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \frac{\partial^2 (\theta_j)^2}{\partial \theta_\mu \partial \theta_n u} = g \sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j \delta_{\mu\nu}\delta_{\mu j} = g a_\mu \delta_{\mu\nu} \sum_{i=\mu}^{n} m_i \tag{3.31}$$

$$\begin{aligned} M_{\mu\nu} &= \frac{\partial^2 T}{\partial \dot{\theta} \mu \partial \dot{\theta}_\mu} = \frac{\partial}{\partial \dot{\theta}_\mu} \sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} a_j \dot{\theta}_j\right) \sum_{l=1}^{i} a_l \frac{\partial \dot{\theta}_l}{\partial \dot{\theta}_\nu} = \sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} a_\mu \delta_{\mu j} \sum_{l=1}^{i} a_\nu \delta_{l\nu}\right) \\ &= a_\mu a_\nu \sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} \delta_{\mu j} \sum_{l=1}^{i} \delta_{l\nu}\right) = a_\mu a_\nu \sum_{i=\max(\mu,\nu)}^{n} m_i \end{aligned} \tag{3.32}$$

From Assumptions 1 and 2, we can simplify (3.31) and (3.32) as

$$K_{\mu\nu} = (n - \mu + 1) m g a \delta_{\mu\nu} \tag{3.33}$$

$$M_{\mu\nu} = (n - \max(\mu,\nu) + 1) m a^2 \tag{3.34}$$

so the characteristic equation for the natural frequencies of small oscillations of the n-pendulum is

$$\boxed{\det((n - \mu + 1) m g a \delta_{\mu\nu} - \omega^2 (n - \max(\mu,\nu) + 1) m a^2) = 0} \tag{3.35}$$

For $n = 1$ this reduces to

$$mga - \omega^2 m a^2 = 0 \rightarrow \omega = \sqrt{g/a}$$

which is just the frequency of small oscillations for the simple pendulum, as expected. For $n = 2$, the double pendulum, we expect 2 natural frequencies to solve the characteristic equation. For an arbitrary value of n, the number of solutions to the $n \times n$ eigenvalue equation will also be n. As we take $n \to \infty$, we will get an infinite number of solutions. We saw this in Section 2.2, where the frequencies were determined by the roots of the zeroth order Bessel function, of which there are infinitely many for this reason.

To find the eigenvalues, then, we use *Mathematica* to solve the eigenvalue equation (3.35) for increasing n. As seen in Figure 3.1, for increasing n, the m^{th} frequency approaches a constant value, exactly corresponding to one half the m^{th} zero of J_0, as predicted by equation (2.16).

The normal modes are the eigenvectors corresponding to the eigenvalues of equation (3.35). Using *Mathematica* we can solve the eigenvalue equation

$$(K - \omega^2 M)v = 0 \tag{3.36}$$

for the eigenvectors v. The i^{th} components of the m^{th} eigenvector v_m correspond to the values of the θ_i coordinate in the m^{th} normal mode. Figure 3.2 shows a plot of the lowest 3 modes for various values of n plotted on top of the corresponding normal mode of the zeroth order Bessel function from equation (2.15).

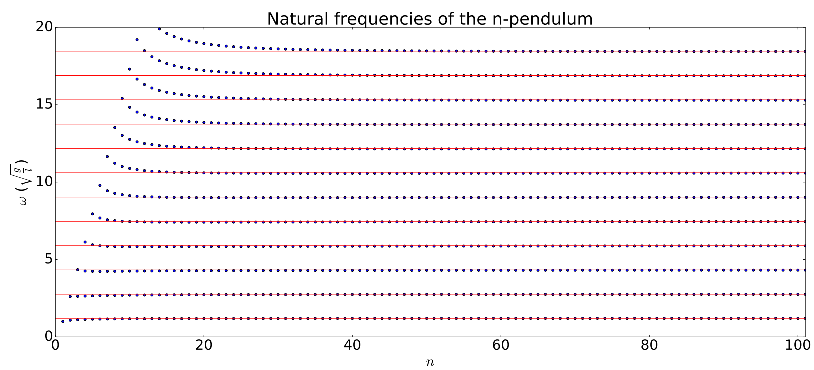

Figure 3.1: The lowest 12 natural frequencies ω of small oscillations of the n-pendulum, in units of $\sqrt{g/\ell}$, for increasing values of n. As $n \to \infty$, the m^{th} natural frequency approaches $\frac{1}{2}z_{0m}$ (denoted by the red horizontal lines), where z_{0m} is the m^{th} positive root of the zeroth order Bessel function J_0.

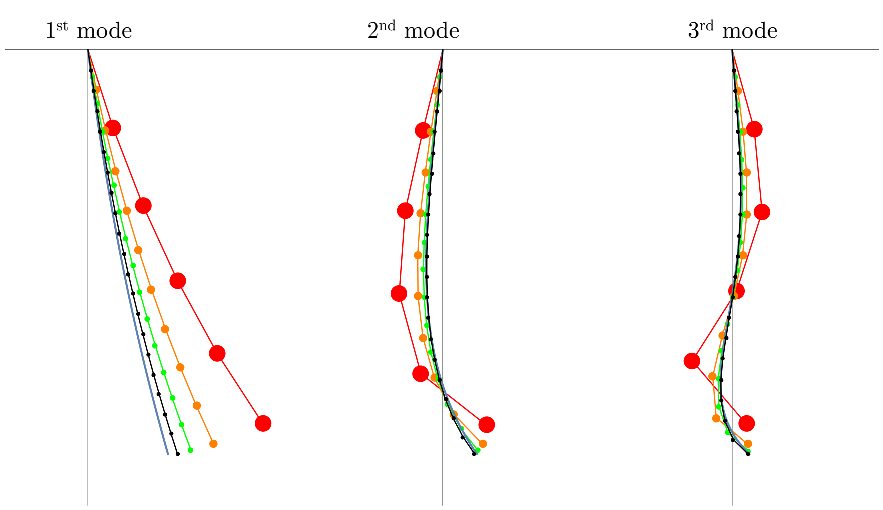

Figure 3.2: The lowest 3 normal modes determined by solving the eigenvalue equation (3.36) for the eigenvectors corresponding to the eigenfrequencies obtained from equation (3.35), for $n = 5$, $n = 10$, $n = 15$, and $\mathbf{n} = \mathbf{20}$ pendulums. The corresponding mode of the zeroth order Bessel function from equation (2.15) is overlaid in blue for comparison. As $n \to \infty$, the continuous case is approached.

3.2 Equations of Motion for a Continuous Mass-Density Rope

The Euler-Lagrange equations of the form presented in the previous section are a set of n differential equations for a system with n degrees of freedom. In solving for the equations of motion for the n-pendulum, we indeed obtained n equations, although our indexing allowed us to write them all in one. However, if we wish to take the limit $n \to \infty$, there will be an infinite number of corresponding Euler-Lagrange equations, and we can't exactly solve an system of infinitely many differential equations. Thus, we must re-derive the Euler-Lagrange equation for the case of a continuous system with infinite number of degrees of freedom.

3.2.1 The Lagrangian Formulation for a Continuous System

We will closely follow the derivation found in chapter 13 of Goldstein's *Classical Mechanics* [2], however the notation will be adjusted to mimic the derivation of the Euler-Lagrange equations presented in class.

For a continuous system, we define the Lagrangian of the system to be

$$L = \iiint \mathcal{L}\ dxdydz \tag{3.37}$$

where $\mathcal{L}$ is called the *Lagrangian density* and is the difference in kinetic energy and potential energy of an infinitesimal piece of the system. For a one-dimensional continuous system, as is our case, let $x = x(y,t)$ be the coordinate scalar functions describing the system. The Lagrangian density then may be a function of the coordinate function $x(y,t)$, the position y or time t, as well as the derivatives with respect to both of these, $\frac{\partial x}{\partial t}$, $\frac{\partial x}{\partial y}$. That is,

$$\mathcal{L} = \mathcal{L}\left(x, \frac{\partial x}{\partial y}, \frac{\partial x}{\partial t}; y, t\right). \tag{3.38}$$

We wish to then extremize the action, defined as the integral over time of the Lagrangian

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2}\int_{y_1}^{y_2} \mathcal{L}\ dydt \tag{3.39}$$

by taking its variation to be zero

$$\delta S = \int_{t_1}^{t_2}\int_{y_1}^{y_2} \left[\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}}\delta\left(\frac{\partial x}{\partial t}\right) + \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial y}}\delta\left(\frac{\partial x}{\partial y}\right)\right] dydt = 0 \tag{3.40}$$

We simplify the second term by integrating by parts in t by letting

$$u = \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}} \qquad v = \delta x$$

$$du = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}} dt \qquad dv = \delta\frac{\partial x}{\partial t}dt = \frac{d}{dt}\delta x dt$$

Thus

$$\int_{t_1}^{t_2}\int_{y_1}^{y_2} \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}}\delta\left(\frac{\partial x}{\partial t}\right) dydt = \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}}\delta x|_1^2 - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}}\delta x dt. \tag{3.41}$$

Similarly, we simplify the third term by integrating by parts in y, letting

$$u = \frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial y}} \qquad v = \delta x$$

$$du = \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial y}} dy \qquad dv = \delta\frac{\partial x}{\partial y}dy = \frac{d}{dy}\delta x dy$$

and thus

$$\int_{t_1}^{t_2}\int_{y_1}^{y_2}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial y}}\delta\left(\frac{\partial x}{\partial y}\right)dydt = \frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial y}}\delta x|_1^2 - \int_{y_1}^{y_2}\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial s}}\delta x ds. \tag{3.42}$$

by requiring the variation in the action to vanish at the endpoints, the boundary terms in (3.41) and (3.42) vanish, and we can write (3.40) as

$$\delta S = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\left[\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial t}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial y}} - \frac{\partial\mathcal{L}}{\partial x}\right]\delta x dydt = 0. \tag{3.43}$$

Since equation (3.43) must be zero for any choice of varied path, the integrand must be zero, and we obtain

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial t}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\frac{\partial x}{\partial y}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \tag{3.44}$$

which is the Euler-Lagrange equation for a one dimensional continuous system. We have reduced our system of equations from infinity to 1!

3.2.2 Equations of Motion Using the Lagrangian Density

We will now apply equation (3.44) to our problem of the hanging rope by determining the Lagrangian density of the system. Consider an infinitesimal chunk of the rope. The arclength of this piece of rope is given by

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2}dy \approx \left(1+\frac{1}{2}\left(\frac{\partial x}{\partial y}\right)^2\right)dy \tag{3.45}$$

where we apply the binomial expansion on the last step for the case of small oscillations. The potential energy[3] stored in this piece of rope is equal to the work done by gravity in moving it by an infinitesimal angle $d\theta$, corresponding to a height change of

$$dh = |ds - ds\cos(d\theta)| = |ds - dy| = \frac{1}{2}\left(\frac{\partial x}{\partial y}\right)^2. \tag{3.46}$$

The work done by gravity is due to the tension force, which as before is due to the weight of all of the rope below the piece in question. So the potential energy of the infinitesimal chunk of rope is, for constant mass density μ,

$$dU = dW = Tdh = \mu(\ell - y)g\frac{1}{2}\left(\frac{\partial x}{\partial y}\right)^2 dy. \tag{3.47}$$

The kinetic energy of the chunk is just

$$dT = \frac{1}{2}dm\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}\mu ds\left(\frac{dx}{dt}\right)^2 \approx \frac{1}{2}\mu dy\left(\frac{dx}{dt}\right)^2. \tag{3.48}$$

where for small oscillations the arclength ds is the same as the infinitesimal length dy to second order. The Lagrangian is the difference in the total kinetic and potential energy of the system,

$$L = \int dT - \int dU = \int_0^{\ell}\left(\frac{1}{2}\mu\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}\mu(\ell - y)g\left(\frac{\partial x}{\partial y}\right)^2\right)dy. \tag{3.49}$$

[3] If we were to simply use the standard $dU = dmgy$ for the potential energy, we would not be taking into consideration the constraint that all the 'particles' are attached together as a rope.

Thus the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\mu\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}\mu g(\ell - y)\left(\frac{\partial x}{\partial y}\right)^2. \tag{3.50}$$

The Euler-Lagrange equations (3.44) for this Lagrangian density are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}} + \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial y}} - \cancelto{0}{\frac{\partial \mathcal{L}}{\partial x}} &= \frac{d}{dt}\left(\mu\frac{\partial x}{\partial t}\right) - \frac{d}{dy}\left(\mu g(\ell - y)\frac{\partial x}{\partial y}\right) = 0 \\
&= \frac{\partial^2 x}{\partial t^2} - g\left((\ell - y)\frac{\partial^2 x}{\partial y^2} - \frac{\partial x}{\partial y}\right) = 0 \\
&\to \boxed{\frac{\partial^2 x}{\partial t^2} = g\left((\ell - y)\frac{\partial^2 x}{\partial y^2} - \frac{\partial x}{\partial y}\right)}
\end{aligned} \tag{3.51}$$

Equation (3.51) is exactly the same as equations (2.7) and (2.9), which we solved by separation of variables for the oscillation of the rope. Thus we have determined the equations of motion both for the continuous rope and for the n-pendulum in the limit $n \to \infty$ from both a Newtonian and Lagrangian Mechanics perspective.

4 Numerical Solution to Nonlinear Equations of Motion for Finite n

As we determined in Section 3.1.1, equation (3.6) gives the full Lagrangian for the n-pendulum:

$$L = \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} a_j\dot{\theta}_j\cos\theta_j\right)^2 + \left(\sum_{j=1}^{i} a_j\dot{\theta}_j\sin\theta_j\right)^2\right] + g\sum_{i=1}^{n}\sum_{j=1}^{i} m_i a_j\cos\theta_j \tag{3.6}$$

The “VariationalMethods” package in *Mathematica* includes the function `EulerEquations`, for which one may input a Lagrangian and list of coordinates and *Mathematica* will determine the resulting Euler-Lagrange equations. The function `NDSolve` may then be used solve the Euler-Lagrange equations numerically, given the boundary conditions, to determine a numerical solution for the motion of each pendulum. The `Animate` function does a nice job of allowing one to watch a movie of the pendulum as it oscillates wildly about its pivot point. For any $n > 1$, the bottommost pendulum will quickly swing wildly about its pivot point. The way in which the system goes berserk is extremely dependent on the initial conditions. In the attached *Mathematica* notebook, one can observe this phenomena for their desired choice of n, although I must warn that the calculation begins to take a significant amount of time for $n > 30$.

To demonstrate the wild behavior of the pendulums, plots of the angular coordinates vs. time for $n = 1, 3, 5$, and 10 can be found in Figure 4.1. All of the plots are created with the initial conditions $\theta_i(t = 0) = \pi/2$ and $\dot{\theta}_i(t = 0) = 0$.

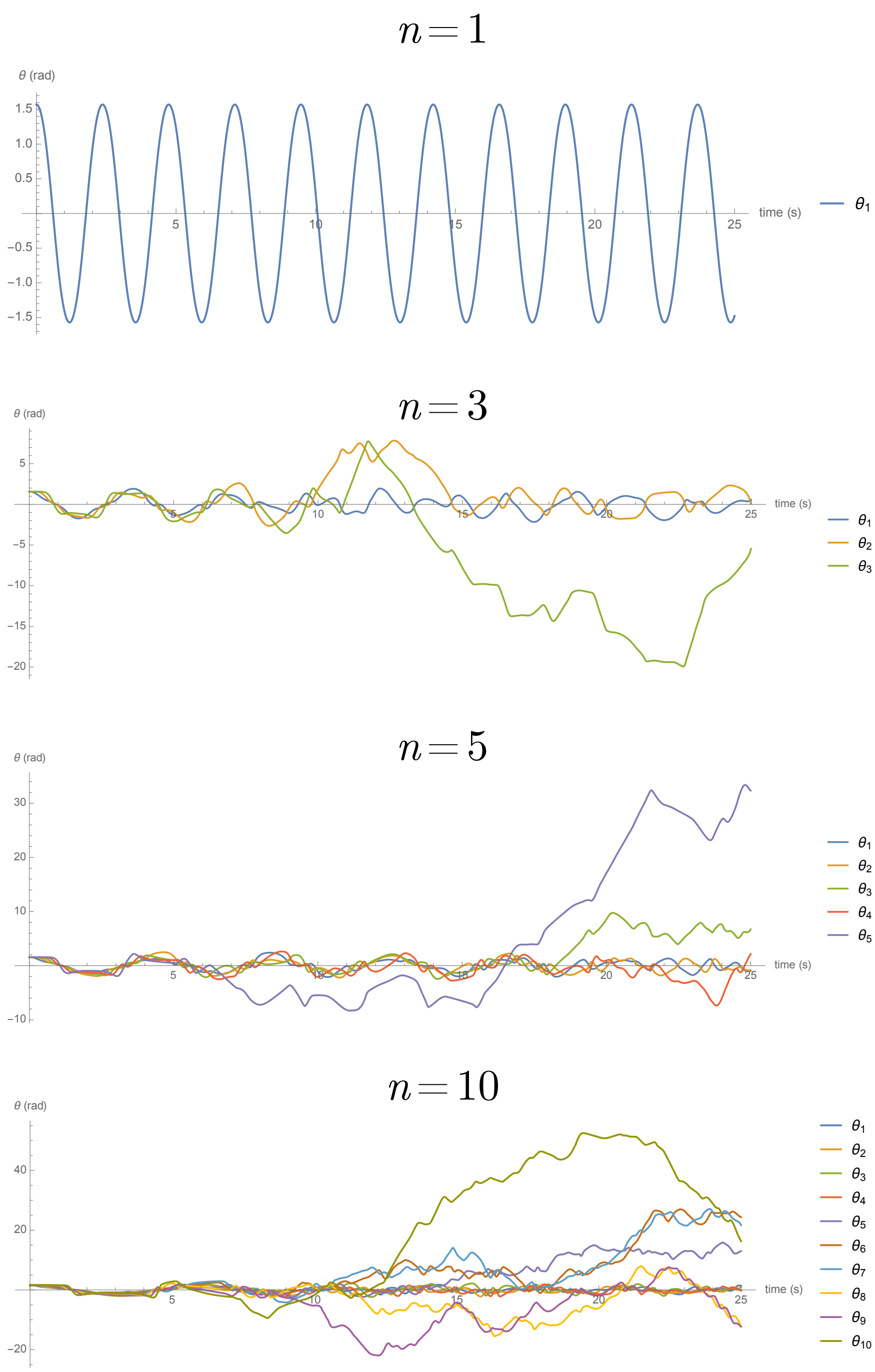

Figure 4.1: Plots of the angular coordinate θ_i for each pendulum bob vs time for (Top) $n = 1$, (Top middle) $n = 3$, (Bottom middle) $n = 5$, and (Bottom) $n = 10$. The chaotic nature of the oscillations becomes apparent around $t = 10$ to 15s.

5 Conclusion

We have extended the analysis for the single, double, and triple pendulum as done in class and in homework to an arbitrary n-pendulum, by approaching the problem both using Newtonian and Lagrangian mechanics. We solved for the equations of motion for small oscillations in the discrete case, and determined the natural frequencies and normal modes of oscillation. In the continuous case, we determined the equation of motion and solved it using the method of separation of variables discovering that the mode shapes are zeroth order Bessel functions and the natural frequencies are multiples of the zeroes of the zeroth order Bessel function J_0. We confirmed this was the case by taking the limit of our solution in the discrete case to obtain a continuous rope and observered that the eigenfrequencies do in fact approach one half the corresponding zero of J_0, and the normal modes also approach the modes corresponding to the Bessel functions in the continuous case. Having solved the small oscillations approximation, we create a *Mathematica* notebook to run a numerical calculation of the nonlinear solution for the full Lagrangian derived for the n-pendulum and make an animation of the motion to observe its chaotic behavior.

Acknowledgments

I would like to thank Gregory Loges and Tanveer Karim for their insight into manipulating and properly re-indexing summations during the calculation in Section 3.1.1.

References

[1] Cannon, J.T.; Dostrovsky, S.; *The Evolution of Dynamics: Vibration Theory from 1687 to 1742*. Springer-Verlag New York Inc, 1981. ISBN: 9781461394631.

[2] Goldstein, H.; Poole, C. P.; Safko, J. L. *Classical Mechanics* (Third Edition). Addison-Wesley, 2001. ISBN: 9780201657029.