Lecture 10  Inductance, Electromagnetic Oscillations, and AC Circuits

Mutual Inductance

Suppose we have two coils next to each other (just like a transformer):

A current \( I_1 \) will create a magnetic fluid in coil 1 \( \Rightarrow \) This in turn creates a magnetic field flux in coil 2.

If \( N_1 \) = \# of coils in 1 and \( N_2 \) = \# of coils in 2, and we say the magnetic flux from 1 to 2 is \( \Phi_{21} \) we have total flux in coil 2 as \( \Phi_{21} \cdot N_2 \)

We can then relate the amount of current in \( I_1 \) and total flux in coil 2 by:

\[
M_{21} = \frac{N_2 \Phi_{21}}{I_1}
\]

where \( M_{21} \) is the mutual inductance units of henry (H); \( 1H = 1V \cdot s = 1A \cdot s \)

If \( I_1 \) is changing then we have an induced emf in coil 2:

\[
\varepsilon_2 = -N_2 \frac{d\Phi_{21}}{dt} = -M_{21} \frac{dI_1}{dt}
\]

\( M_{21} \) only depends on geometry of system (# of turns, area of loops) (current too but mostly geometry)
Suppose we reverse the system, coil 1 is off and $I_2$ current runs in coil 3.

Total flux on 1 is then $N_1 \Phi_{1a} \Rightarrow M_{12} = \frac{N_1 \Phi_{1a}}{I_2}$

\[ E_1 = -N_1 \frac{d\Phi_{1a}}{dt} = -M_{12} \frac{dI_2}{dt} \]

It can be proven (but not now) that $M_{12} = M_{21} = M$

\[ E_1 = -M \frac{dI_2}{dt}; \quad E_2 = -M \frac{dI_1}{dt} \]

**Self Inductance** Suppose we had just a single group of $N$ loops (like a solenoid)

Say $I_1$ goes through the coil, but $I_1$ is changing so a changing magnetic field is created inside the solenoid.

The changing magnetic field will create an induced emf and therefore induced current.

* For positive changing magnetic flux the Ind will go against $I_1$ (maintain status quo)
* For negative changing magnetic flux the Ind will go with $I_1$

If the solenoid has $N$ turns and the flux it creates is $\Phi_3$, we can relate the total flux in the solenoid ($N_3 \Phi_3$) with the current $I$ by:
\[ L = \frac{N \Phi_B}{I} \text{ where } L \text{ is the self-inductance. [units of H]} \]

Then we can write the induced emf.

\[ \varepsilon = -N \frac{d\Phi_B}{dt} = -L \frac{dI}{dt} \]

We can usually put inductors in circuits (symbol \( \text{in} \)). Typical inductors range from 1 uH to 1H.

An inductor acts almost like a resistor (it slows down or impedes current flow). We will expand more on this later.

Ex: Solenoid self-inductance: For solenoid \( B = \frac{NI\mu_0}{l} \)

\[ N = 100 \quad L = \frac{N \Phi_B}{I} = \frac{N^3\mu_0IA}{Il} = \frac{N^3A\mu_0}{I} \]

\[ I = 5.0 \text{ cm} \]

\[ A = 0.30 \text{ cm}^2 \]

\[ L = \left(4\pi \times 10^{-7} \text{T.m/A}\right) (100)(3 \times 10^5 \mu_0) \left(5 \times 10^{-3} \text{m} \right) = 7.5 \mu\text{H} \]

Ex: Coaxial Cable inductance (like capacitance) we need to integrate to find flux here.

First we need to find \( B \) for inner region.
Using Ampere's Law: \( \oint B \cdot dl = \mu_0 I_{nc} \)

\[ B = \frac{\mu_0 I}{2\pi r} \quad \text{for} \quad r < r_2 \]

\[ B = 0 \quad \text{outside cable} \quad I_{nc} = 0 \]

Where is \( dA \) here? \( \Rightarrow \text{not cross-sectional area, b/c} \quad B \cdot dA = 0 \)

It is the longitudinal cross section.

\[ \Phi_B = \int B \cdot dA \]

\[ \Phi_B = \frac{\mu_0 I}{2\pi} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \]

so we have flux \( \Phi_B \) would give \( \Phi_B = 0 \)

\[ L = \frac{\Phi_B}{I} = \frac{\mu_0 L}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \]
Energy Stored in a Magnetic Field

In a coil in an inductor so we have changing power:

\[ P = I \mathcal{E} = LI \frac{dI}{dt} \]

\[ dW = PDt = LI dI \]

\[ W = \int_0^L LI dI = \frac{1}{2} L I^2 \rightarrow \text{very similar to capacitors.} \]

\[ U = \frac{1}{2} LI^2 \]

For a capacitor energy is stored in the electric field; for an inductor energy is stored in the magnetic field.

For a solenoid only:

\[ L = \frac{\mu_0 N^2 A}{l} \quad \text{and} \quad B = \frac{\mu_0 NI}{l} \]

\[ U = \frac{1}{2} LI^2 = \frac{1}{2} \left( \frac{\mu_0 N^2 A}{l} \right) \left( \frac{B l}{\mu_0 N} \right)^2 = \frac{1}{2} \frac{B^2 A l}{\mu_0} \text{ Volume of Solenoid} \]

The energy density:

\[ \frac{U}{\text{Volume}} = \frac{1}{2} \frac{B^2}{\mu_0} = \mathcal{U} \]
LR Circuits

Say the switch is initially in position 1 ⇒ we are "charging" the inductor.

Once in switch 0 position our instantly changing current in the inductor stirs to create an emf (that travels against the usual current flow).

So using Kirchhoff:

\[ V_0 - IR - L \frac{dI}{dt} = 0 \]

\[ V_0 = L \frac{dI}{dt} + RI \]

If we want to loop at current we can rewrite this as:

\[ \int_{I_0}^{I} \frac{dI}{V_0 - IR} = \int_{0}^{t} \frac{dt}{L} \]

\[ -\frac{1}{R} \ln \left( \frac{V_0 - IR}{V_0} \right) = \frac{t}{L} \]

\[ I = \frac{V_0}{R} \left( 1 - e^{-\frac{t}{\tau}} \right) \text{ where } \tau = \frac{L}{R} \Rightarrow \text{ time constant of LR circuit} \]

\( \tau \Rightarrow \text{ time when } I \text{ has reached 63% of its max value.} \)

\[ I \uparrow - \quad I_{\max} = \frac{V_0}{R} \]

\[ \rightarrow 0.63 I_{\max} \]
what about position 2?

Once at a max current in the inductor we go to switch $\Theta \rightarrow$ now induced emf acts like a battery in the circuit, we have:

$$L \frac{dI}{dt} + RI = 0$$

$$\int \frac{I}{I_0} \frac{dI}{I} = - \int \frac{R}{L} dt$$

$$\ln \left( \frac{I}{I_0} \right) = - \frac{R}{L} t \Rightarrow I = I_0 e^{-\frac{R}{L} t}$$

LC Circuits & Electromagnetic oscillations

The capacitor here acts like a battery with decreasing voltage (we went this b/c that way $I$ is decreasing)

$$\Rightarrow -L \frac{dI}{dt} + \frac{Q}{C} = 0 \Rightarrow I = \frac{dQ}{dt} \text{ since } I \text{ is decreasing}$$

$$+ \frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$
\[
\frac{d^2 Q}{dt^2} = -\frac{Q}{LC}
\]

As a guess, \( Q = Q_0 \cos(wt + \phi) \) where \( Q_0 \) and \( \phi \) depend on initial conditions.

Plug into above equation:

\[-w^2 Q_0 \cos(wt + \phi) + \frac{1}{LC} Q \cos(wt + \phi) = 0\]

\[\Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega = 2\pi f = \sqrt{\frac{1}{LC}}\]

Charge is oscillating here! \( Q = Q_0 \cos(wt + \phi) \)

\[I = \frac{dQ}{dt} = \omega Q_0 \sin(wt + \phi)\]

\[= I_0 \sin(wt + \phi) \Rightarrow \text{current oscillates in circuit.}\]

Let's look at energy

\[
U_E = \frac{1}{2} \frac{Q_0^2}{C} \cos^2(wt + \phi)
\]

\[
U_B = \frac{1}{2} LI^2 = \frac{Lw^2 Q_0^2 \cos^2(wt + \phi)}{2}
\]

\[= \text{energy oscillates between } B \text{ and } C.\]

\[U = U_E + U_B = \frac{Q_0^2}{C}\]
Using Kirchhoff's Rule:

\[ -\frac{LdI}{dt} - IR + \frac{Q}{C} = 0 \]

Initial current, \( I = -\frac{dQ}{dt} \)

\[ \Rightarrow \frac{Ld^2Q}{dt^2} + Rd\frac{dQ}{dt} + \frac{1}{C}Q = 0 \]  looks like damped harmonic oscillator.

The solution (you should have done this in mechanics) is:

\[ Q = Q_0 e^{-\frac{R}{2L}t} \cos(\omega' t + \phi) \]

where \( \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \)

The system is:
- critical damped \( R^2 = 4L/C \)
- underdamped \( R^2 < 4L/C \)
- overdamped \( R^2 > 4L/C \)
AC Circuits

Produces current equal to
\[ I = I_0 \cos(\omega t) \]

Remember here that:
\[ V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \]
\[ I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \]

For a resistor \( V = IR \Rightarrow V = \frac{I_0 R \cos(\omega t)}{V_0} \) current and voltage are in phase.

Inductor:
\[ V = L \frac{dI}{dt} = -\frac{\omega LI_0 \sin(\omega t)}{V_0} \]

The minus means they are out of phase.

Using \( \sin \theta = -\cos(\theta + 90^\circ) \)

\[ V = \omega LI_0 \cos(\omega t + 90^\circ) \]

This our phase.

\[ V_0 = \omega LI_0 = \times I_0 \] (like \( V = I R \))

\( \times \) is called our inductive reactance (or impedance) (reduces current flow).
Capacitor \[ \frac{dQ}{dt} = \frac{d}{dt} \int_0^t V \cos \omega t \ dt \]

\[ V = \frac{q}{C} \Rightarrow Q = \int_0^t \int_0^t V \cos \omega t \ dt \]

\[ Q = \frac{I_0}{\omega} \sin \omega t \]

\[ V = I_0 \left( \frac{1}{\omega C} \right) \sin \omega t = \frac{I_0}{\omega C} \cos(\omega t - 90^\circ) \]

\[ \frac{1}{\omega C} \text{ is capacitive reactance (or impedance)} \]

These impedances are only for A.C. circuits.

Suppose we have this circuit:

\[ A \quad \text{and} \quad B \quad \text{are high pass filter} \]