Capacitance, Dielectrics & Energy Storage

An arrangement of charged conductors is called a capacitor; there are three main arrangements:
1) two sheets or plates
2) concentric spheres
3) two cylinders

Many applications in electronics, automobile ignition systems, etc.

A capacitor has a certain measure of capacitance defined as:

\[ C = \frac{Q_{\text{stored}}}{V} \]

where \( V \) is the potential difference between two charge conductors.

Units \( \left[ \frac{C}{V} \right] = \text{Farad (F)} \)

Typical capacitors:

\( 1 \text{pF} \rightarrow 10^3 \text{mF} \)

What is stored charge? Suppose we connect two conducting plates of area \( A \) to some battery.

+Q -Q

Electrons within the left conductor are going to see the + side of the battery and hurry towards it, creating an overall +Q on the left plate. This induces a +Q on the opposite plate (+ charges head away to - side of battery).
We take away the wire and we are left with two conducting plates of area A each with equal and opposite charges Q. This Q is defined as the "stored charge".

Initially our plates were neutral now they each have a stored charge Q.

So \[ C = \frac{Q \text{ stored}}{V} \]

For two conductors to calculate capacitance we always integrate from \( \Theta \to \Theta \) set our left plate at origin

\[ V = V_L - V_R = - \int_{R}^{E} Ed \cos \Theta \]

When calculating capacitance we integrate anti parallel to the electric field.

So now \( \Theta = \pi \Rightarrow \cos \Theta = -1 \)

The only line which that our capacitance will be positive.

\[ V = \int_{R}^{L} Ed \leq E(L - R) = Ed \]

Between two plates we know \( E = \frac{V}{\epsilon_0} = \frac{Q}{A \epsilon_0} \)

\[ V = \frac{Qd}{A \epsilon_0} \]
\[ C = \frac{Q \Delta \varepsilon_0}{d} = \frac{A \varepsilon_0}{d} \Rightarrow \text{capacitance only depends on the area of plates and distance between them!} \]

We will see that \( C \) is always a constant of the system!

**Spherical Capacitor**

Consider two concentric thin, uniformly charged conductive shells of radius \( r_a \) and \( r_b \)

\[ +Q \text{ inside } -Q \text{ outside } \Rightarrow \text{stored is } Q \]

we need potential difference \( r_b \) and \( r_a \)

we know from Gauss' law that \( E = \frac{kQ}{r^2} \) with \( r_b < r_a \)

\[ V = V_b - V_r \Rightarrow \int \frac{V_{\text{left}}}{r_b} \cos \pi \left[ \cos \pi - 1 \right] \, dr \]

\[ V = kQ \left( \frac{1}{r_b} - \frac{1}{r_a} \right) \]

\[ \Rightarrow C = \frac{Q}{V} = \frac{1}{k \left( \frac{1}{r_b} - \frac{1}{r_a} \right)} \Rightarrow \text{doesn't depend on charge only geometry of spheres!} \]

**Cylindrical Capacitor** Suppose we have a cylindrical shell at \( r_a \) and an interior wire of radius \( r_b \) located at the origin.

we assume that \( r_b << 1 \neq 0 \)

Suppose of cylindrical capacitor has a length \( l \)
Same thing as before:

stored charge is \( Q \), if we go from \( \Theta \) to \( \Theta \)

\[
V = V_+ - V_- = - \int_{r_a}^{r_b} E \, dl \cos \Theta
\]
\( dl \) is inward \( \Rightarrow dl = -dr \); \( \cos \Theta = -1 \)

Since \( r_b < \Theta \) we can treat it like a wire:

\[
E = \frac{2\pi k}{r} \quad \text{radially away from wire.}
\]

\[
V = - \frac{2\pi k}{l} \int_{r_a}^{r_b} \frac{dr}{r} = -\frac{2\pi kQ}{e} \left( \ln \left( \frac{r_a}{r_b} \right) \right)
\]

\[
C = \frac{Q}{V} = \frac{e}{2\pi k \ln \left( \frac{r_a}{r_b} \right)}
\]

Two long parallel wires of radius \( R \), \( R \ll \Theta \)

with \( \Theta \) at origin

---

\[
E = \frac{2\pi k}{x} \quad \text{from a wire}
\]

\[
\text{so} \quad E_+ = \frac{2\pi k}{x} \quad ; \quad E_+ = \frac{2\pi k}{(d-x)}
\]

Using superposition:

\[
E = \frac{2\pi k}{x} \left( \frac{1}{x} + \frac{1}{d-x} \right) \quad \text{to the left}
\]
We integrate from outside one wire to another.

\[ V = V_+ - V_- = - \int_{d-R}^{R} \varepsilon d x \cos \pi \]

\[ V = 2k \lambda \int_{d-R}^{R} \left( \frac{1}{x} + \frac{1}{d-x} \right) d x \]

\[ V = 2k \lambda \left( \ln \left( \frac{d-R}{R} \right) - \ln \left( \frac{R}{d-R} \right) \right) \]

\[ V = k \lambda \ln \left( \frac{d-R}{R} \right) \quad \text{say} \quad \lambda = \frac{Q}{l} \]

\[ \left[ C = \frac{Q}{V} = \frac{l}{k \ln \left( \frac{d-R}{R} \right)} \right] \quad l \text{is length of wire} \]
Electric Energy storage

How much energy is stored in a capacitor when charging up to $Q$ or $-Q$?

(Work done as we increase/decrease $Q$)

The work needed to store a charge $Q$ is:

$$ W = \int_0^Q V \, dq $$

we integrate from $0 \to Q$

work is $+b/c$ we are doing it on the system

$$ W = \int_0^Q V \, dq = \frac{1}{c} \int_0^Q E \, dq = \frac{1}{c} \frac{Q^2}{2} $$

The total energy stored is then (energy to go from net $0$)

$$ \Delta U = W = \frac{1}{2} \frac{Q^2}{c} $$

known that $C = \frac{Q}{V}$

$$ \Delta U = \frac{1}{2} Q V = \frac{1}{2} C V^2 $$

Assume we have two plates:

$$ U = \frac{1}{2} CV^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d} (E_0 d^2) $$

$$ U = \frac{1}{2} \varepsilon_0 E_0 A d $$

Volume between plates

$$ \frac{U}{\text{vol}} = \frac{1}{2} \varepsilon_0 E^2 = \frac{U}{\text{energy density}} $$

electric energy per unit volume in any region inside capacitor is $\propto$ to square of $E_0$ field
Dielectrics

* Most capacitors are filled with an insulator such as plastic, paper, etc.

The insulator allows charge to travel less readily through between the plates (air is not ionized). This allows ability for very high V.

Also good to keep structurally keep plates close to each other (decreasing \( d \) \( \Rightarrow \) increase \( C \)).

The addition of a dielectric increases the capacitance by a dielectric constant \( k \) (as lecture I referred to this as \( \varepsilon_0 \))

\[
C = k \cdot \varepsilon_0
\]

\( \varepsilon_0 \) w/o dielectric

<table>
<thead>
<tr>
<th>Material</th>
<th>Constant ( \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vac.</td>
<td>( 1 )</td>
</tr>
<tr>
<td>Air</td>
<td>( 1.0006 )</td>
</tr>
<tr>
<td>Paper</td>
<td>( 3.7 )</td>
</tr>
<tr>
<td>Oil</td>
<td>( 4 )</td>
</tr>
<tr>
<td>Porcelain</td>
<td>( 60 - 8 )</td>
</tr>
</tbody>
</table>

\( \Rightarrow \) \( k \) is material specific

For two plates:

\[
C = \frac{k \varepsilon_0 A}{d}
\]

\( \Rightarrow U = \frac{1}{2} k \varepsilon_0 E^2 = \frac{1}{2} \varepsilon E^2 \)
Dielectric & electric field:

\[ C = kC_0 = k \frac{Q}{V_0} = \frac{V}{V'} \]

\[ V' = V_0 \text{ voltage without dielectric} \]

Between two plates \( V' Ed = E = \frac{V_0}{kd} = \frac{E_0}{k} \)

The electric field is decreased with the addition of a dielectric.
Capacitors in Series and Parallel Circuits

Parallel symbol for capacitor is \( \parallel \).

Three capacitors connected in parallel exist as shown.

\[ V_{\text{batt}} = V \]

For capacitors connected in parallel, \( V_{\text{batt}} = V \) exists across all of them. The stored charge on each is then:

\[ Q_1 = C_1 V \]
\[ Q_2 = C_2 V \]
\[ Q_3 = C_3 V \]

The total stored charge is then:

\[ Q_{\text{tot}} = Q_1 + Q_2 + Q_3 \]

\[ Q_{\text{tot}} = C_{\text{eq}} V = (C_1 + C_2 + C_3) V \]

The equivalent capacitance. We can replace capacitors in parallel with a "different" capacitor with capacitance \( C_{\text{eq}} \).
Series

Series are connected such as:

\[
\begin{align*}
+V - \Delta V - \Delta V - \Delta V - +V \\
C_1 & \quad C_2 & \quad C_3 \\
\text{V}_{\text{batt}} & \quad + \quad \text{V}_{\text{batt}} = V
\end{align*}
\]

The charge induced on capacitors is the same because they are in series, so like a big capacitor that induces a stored charge.

For capacitors in series \( V_{\text{batt}} = V = V_1 + V_2 + V_3 \) if addition of voltages across the capacitors. But we know that:

\[
\begin{align*}
Q &= C_1 V_1, & V_1 &= \frac{Q}{C_1} \\
Q &= C_2 V_2, & V_2 &= \frac{Q}{C_2} \\
Q &= C_3 V_3, & V_3 &= \frac{Q}{C_3}
\end{align*}
\]

Using the above equations:

\[
V_5 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} = \frac{Q}{C_{\text{eq}}}
\]

\[
\frac{1}{C_{\text{eq}}} = \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right) \quad \text{C}_{\text{eq}} \text{ is smaller than } C_1, C_2, C_3 \quad \text{replace single capacitor with}
\]

Say: \( C_1 = C_0 = C_3 = C \)

What is capacitance of diagram to left?

Looking at \( C_2 \) \& \( C_3 \) are parallel so

\[
C_{23} = C_2 + C_3 = 2C
\]

Then \( C_{23} \) \& \( C_1 \) are in series:

\[
\frac{1}{C_{\text{eq}}} = \frac{1}{2C} + \frac{1}{C} = \frac{3}{2C} \quad \Rightarrow C_{\text{eq}} = \frac{3}{2} C
\]