Solving ablative RT Problem in AstroBEAR

1.Quasi-equilibrium

The quasi-equilibrium in RT problem is defined by the balance of the thermal pressure, Ram pressure and the gravity force. In equilibrium, the Euler equations are:

$$\frac{\partial \rho v}{\partial y}=0$$

$$\rho v \frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}-\rho g$$

$$\rho v C_v \frac{\partial T}{\partial y}=-p\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}(\kappa T^n \frac{\partial T}{\partial y})$$

from the first equation we know that $\rho v=\rho_0 v_0$, which is just the outflow at the bottom. If we define the heat flux as:

$$q=\kappa T^n \frac{\partial T}{\partial y}$$

we can then write the equations as:

$$\frac{\partial \rho}{\partial y}=\frac{\frac{\rho R q}{\kappa T^n}+\rho g}{\frac{\rho_0^2 v_0^2}{\rho^2}-RT}$$

$$\frac{\partial T}{\partial y}=\frac{q}{\kappa T^n}$$

$$\frac{\partial q}{\partial y}=\frac{\rho_0 v_0 C_v q}{\kappa T^n}-\rho_0 v_0 RT\frac{\frac{Rq}{\kappa T^n}+g}{\frac{\rho_0^2 v_0^2}{\rho^2}-RT}$$

with ρ , v , q known on the two boundaries.

Using parameters (in SI units, Temperature is measured in Joule):

R = 4.7904e+26

C_v=7.186e+26

κ=3.734e+69

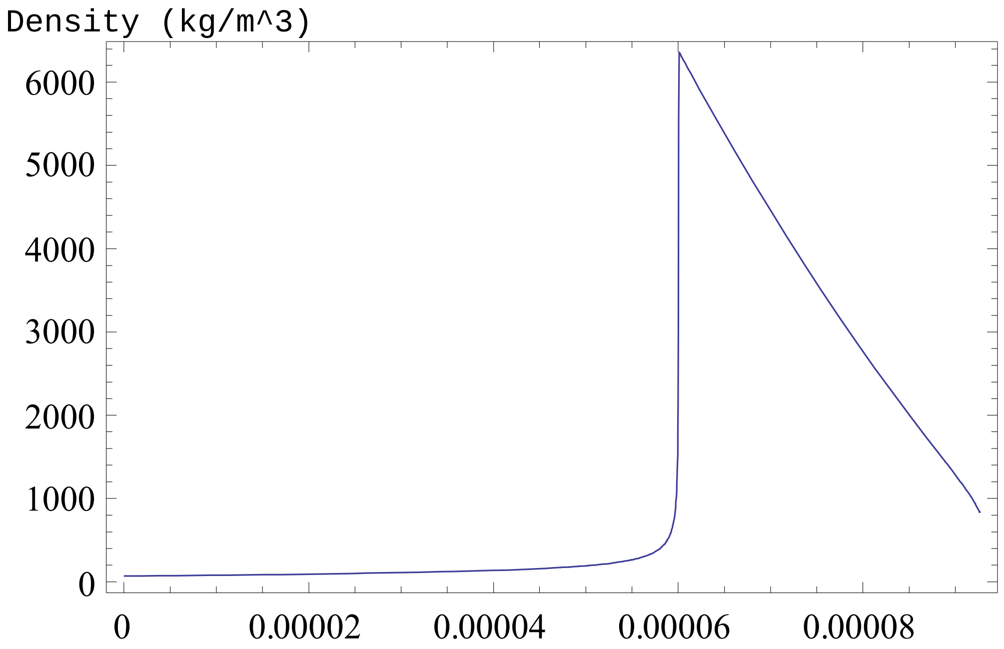
g = 1.0e+14

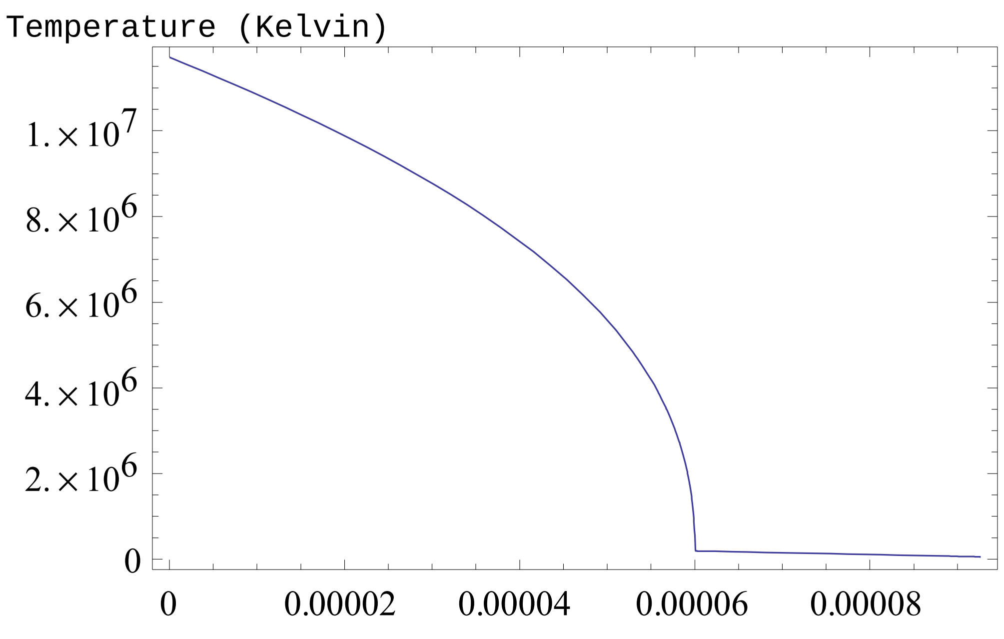
q = -5.876e+18

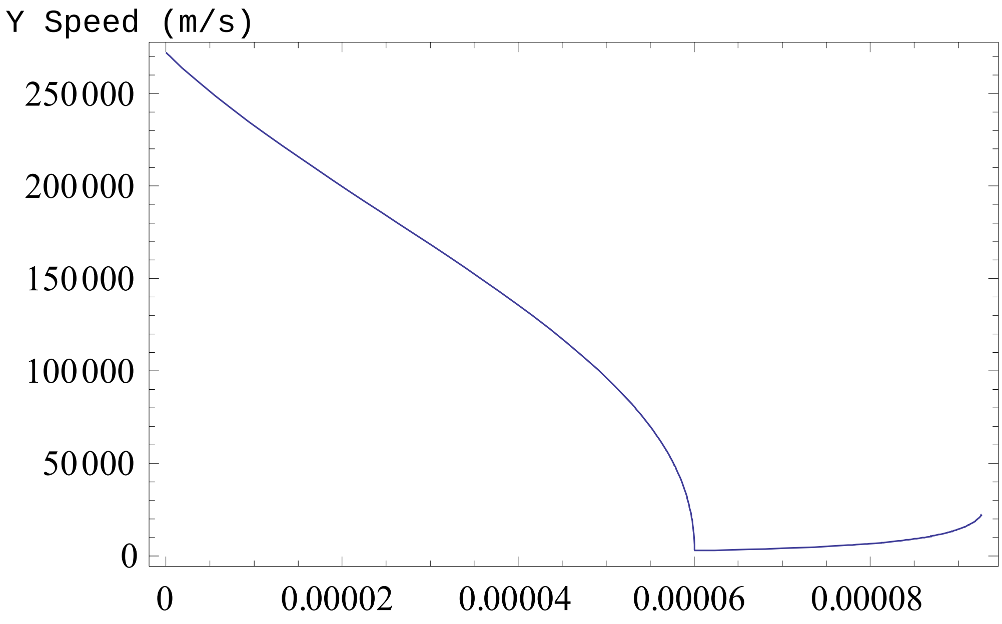
ρ_0=68.1622919147237

v_0=−272144.604867564

The solutions to this equation set are posted below:


Density (kg/m^3)
6000
5000
4000
3000
2000
1000
0
0
0.00002
0.00004
0.00006
0.00008


Temperature (Kelvin)
$1.\times10^{7}$
$8.\times10^{6}$
$6.\times10^{6}$
$4.\times10^{6}$
$2.\times10^{6}$
0
0
0.00002
0.00004
0.00006
0.00008



2. Euler Fluid Equations with diffusion in c.g.s.

The Euler equations can be written as:

$$\rho_t+(\rho u)_x+(\rho v)_y=0$$
$$(\rho u)_t+(\rho u^2+p)_x+(\rho u v)_y=0$$
$$(\rho v)_t+(\rho u v)_x+(\rho v^2+p)_y=-\rho g$$
$$\left(\frac{\rho v_t^2}{2}+\frac{p}{\gamma-1}\right)_t+\nabla\bullet\left[v_t\left(\frac{\rho v_t^2}{2}+\frac{\gamma p}{\gamma-1}\right)\right]=\nabla\bullet(\kappa T^n\nabla T)$$

From the last equation, we have:

$$\left(\frac{\rho v_t^2}{2}\right)_t+\left(\frac{p}{\gamma-1}\right)_t+\left(\frac{\rho v_t^2}{2}+\frac{\gamma p}{\gamma-1}\right)\nabla\bullet v_t+v_t\bullet\nabla\left(\frac{\rho v_t^2}{2}+\frac{\gamma p}{\gamma-1}\right)=\nabla\bullet(\kappa T^n\nabla T)$$

The second term can be written as:

$$\frac{p}{\gamma-1}=\frac{(1+Z)\rho k_B T}{(\gamma-1)m_i}$$

If we write $c_v=\dfrac{(1+Z)}{(\gamma-1)m_i}$, and measure T in terms of energy, then we have the EOS:

$$\frac{p}{\gamma-1}=c_v\rho T$$

then the energy equation can be written as:

$$\left(\frac{\rho v_t^2}{2}\right)_t+(c_v\rho T)_t+\left(\frac{\rho v_t^2}{2}+\gamma c_v\rho T\right)\nabla\bullet v_t+v_t\bullet\nabla\left(\frac{\rho v_t^2}{2}+\gamma c_v\rho T\right)=\nabla\bullet(\kappa T^n\nabla T)$$

Simplify this equation by using the momentum and mass conservations, we get the energy equation in the form of:

$$\rho c_v(\partial_t T+v\bullet\nabla T)=-p\nabla\bullet v+\nabla\bullet(\kappa T^n\nabla T)$$

Notice that in this form, all quantities are in SI with T measured in Joule. So it's actually:

$$\rho^S c_v^S(\partial_t k^S T^S+v^S\bullet\nabla k^S T^S)=-p^S\nabla^S\bullet v^S+\nabla^S\bullet(\kappa^S(k^S)^n(T^S)^n\nabla^S k^s T^S)$$

To convert to the c.g.s units, we have the following conversion factors:

$$\rho^S=1000\,\rho^G \qquad p^S=1/10\;p^G$$
$$v^S=1/100\,v^G \qquad \nabla^S=100\,\nabla^G$$
$$c_v^S=1000\,c_v^G \qquad k^S=10^{-7}k^G$$

Substitute, we find that:

$$\rho^G c_v^G(\partial_t k^G T^G+v^G\bullet\nabla k^G T^G)=-p^G\nabla^G\bullet v^G+\nabla^G\bullet(10^{(-7n-2)}\kappa^S(k^G)^n(T^G)^n\nabla^G k^G T^G)$$

If we define $\kappa^G=10^{(-7n-2)}\kappa^S$, then:

$$\rho^G c_v^G(\partial_t k^G T^G+v^G\bullet\nabla k^G T^G)=-p^G\nabla^G\bullet v^G+\nabla^G\bullet(\kappa^G(k^G)^n(T^G)^n\nabla^G k^G T^G)$$

Using operator splitting method, we need to solve:

$$\rho c_v^G(\partial_t k^G T)=\nabla^G\bullet(\kappa^G(k^G)^n T^n\nabla k^G T)$$

Divide by k and cv in c.g.s., we get:

$$\rho(\partial_t T)=\nabla\bullet(\frac{\kappa^G(k^G)^n}{c_v^G}T^n\nabla T)$$

define:

$$\kappa_0=\frac{\kappa^G(k^G)^n}{c_v^G}=\frac{10^{(-7n+1)}(k^G)^n\kappa^S}{c_v^S}$$

then we have the equation ready to be scaled:

$$\rho(\partial_t T)=\nabla\bullet(\kappa_0 T^n\nabla T)$$

For the flux equation, we have:

$$q^S=\kappa^S(k^S)^n T^n\nabla^S T$$

we need a form like:

$$q_0=\kappa_0 T^n\nabla^G T$$

we can divide these two equations and use the relations between SI units and Gauss units. The result is:

$$q_0=\frac{q^S}{10\,c_v^S k^S}$$

The scales can be obtained by writing equation $\rho(\partial_t T)=\nabla\bullet(\kappa_0 T^n\nabla T)$ in the forms of scalings:

$$\frac{rscale\,Tscale}{tscale}=\frac{\kappa\,scale\,Tscale^{(n+1)}}{lscale^2}$$

simplify this form, we get:

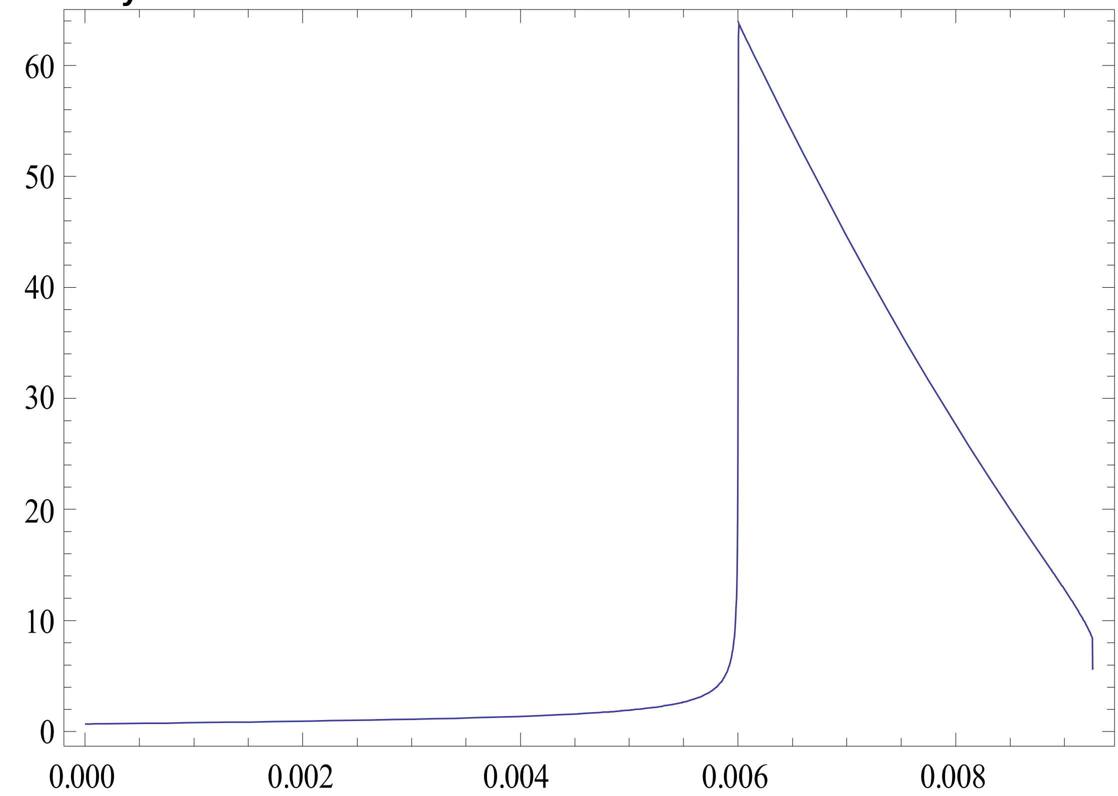
$$\kappa\,scale=\frac{lscale\,rscale^{1/2}\,pscale^{1/2}}{Tscale^n}$$

also, we have:

$$qscale=\frac{\kappa\,scale\,Tscale^{n+1}}{lscale}=rscale^{1/2}\,pscale^{1/2}\,Tscale$$

Using the rescaled parameters, the Euler equation is solved and the equilibrium is plotted as follows:

Density



Temperature