This false color image is taken from Dan Howell's experiments. This is a 2D experiment in which a collection of disks undergoes steady shearing. The red regions mean large local force, and the blue regions mean weak local force. The stress chains show in red. The key point is that on at least the scale of this experiment, forces in granular systems are inhomogeneous and intermittent if the system is deformed. We detect the forces by means of photoelasticity: when the grains deform, they rotate the polarization of light passing through them.

Critical Scaling of a Sheared Granular Material at the Jamming Transition

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• introduction - the Jamming Phase Diagram

• scaling ansatz for jamming at $T = 0$

• simulations in 2D at finite fixed shear strain rate
  large system with $N = 65,536$ particles
  critical exponents for viscosity and yield stress

• quasi-static simulations in 2D of slowly sheared system
  finite size scaling with $N = 64$ to $1024$ particles
  critical exponents for yield stress, correlation length, energy, pressure

• conclusions
"point $J$" is a critical point like in equilibrium transitions.

Critical scaling at point $J$ influences behavior at finite $T$ and finite $\sigma$.

Understanding $T = 0$ jamming at "point $J$" may have implications for understanding the glass transition at finite $T$. 
jamming phase diagram $T = 0$ (Olsson & Teitel PRL 2007)

analogy to Ising model

shear strain rate $\dot{\gamma} \leftrightarrow h$

shear stress $\sigma \leftrightarrow m$

yield stress $\sigma_Y \sim (\phi - \phi_J)^\Delta$

Hatano, J Phys Soc Jpn 2008
Scaling ansatz for jamming rheology

Control parameters: \( \delta \phi \equiv \phi - \phi_J, \quad \dot{\gamma}, \quad L^{-1} \quad (L = \sqrt{N}) \)

Critical point \( J \): \( \delta \phi = \dot{\gamma} = L^{-1} = 0 \)

\( N \) = number particles

**Scaling hypothesis (2nd order phase transitions):**
At a critical point, all quantities that vanish or diverge do so as some power of a diverging correlation length \( \xi \). When \( \xi \to b\xi \), these quantities scale as a power of \( b \), where \( b \) is an arbitrary length rescaling factor.

\[
\delta \phi \sim b^{-1/\nu}, \quad \dot{\gamma} \sim b^{-z}, \quad L \sim b, \quad \sigma \sim b^{-\Delta/\nu}
\]

\( \nu \) is correlation length critical exponent
\( z \) is dynamic critical exponent
\( \Delta \) is yield stress exponent

\[
\sigma(\phi, \dot{\gamma}, L) b^{\Delta/\nu} = f(\delta \phi b^{1/\nu}, \dot{\gamma} b^z, L^{-1} b)
\]
scaling law

\[
\sigma(\phi, \dot{\gamma}, L) = b^{-\Delta/\nu} f(\delta \phi b^{1/\nu}, \dot{\gamma} b^z, L^{-1} b)
\]

**Thermodynamic limit:** \( L \to \infty \)

choose \( b = |\delta \phi|^{-\nu} \) \( \Rightarrow \) \( \sigma = |\delta \phi|^\Delta f(\pm 1, \dot{\gamma} |\delta \phi|^{-z\nu}, 0) \)

\( \dot{\gamma} = 0 \) \( \Rightarrow \) \( \sigma_Y \sim \delta \phi^\Delta \)

choose \( b = \dot{\gamma}^{-1/z} \) \( \Rightarrow \)

\[
\sigma = \dot{\gamma}^{\Delta/z\nu} f(\delta \phi \dot{\gamma}^{-1/z\nu}, 1, 0)
\]

when plot \( \frac{\sigma}{\dot{\gamma}^{\Delta/z\nu}} \) vs \( \frac{\delta \phi}{\dot{\gamma}^{1/z\nu}} \)

\( \sigma(\phi, \dot{\gamma}) \) data should collapse to a single scaling curve

Olsson & Teitel PRL 2007
scaling of shear viscosity

\[ \sigma = |\delta \phi|^\Delta f(\pm 1, \dot{\gamma} |\delta \phi|^{-z\nu}) \]

\[ \eta \equiv \frac{\sigma}{\dot{\gamma}} = |\delta \phi|^{\Delta-z\nu} g(\pm 1, \dot{\gamma} |\delta \phi|^{-z\nu}) \]

viscosity exponent \( \beta \)

as \( \dot{\gamma} \to 0, \eta \sim |\delta \phi|^{-\beta} \Rightarrow \beta = z\nu - \Delta \)

\[ z\nu = \Delta + \beta \]

\[ \sigma = \dot{\gamma}^{\Delta/z\nu} f(\delta \phi \dot{\gamma}^{-1/z\nu}, 1) \]

\[ \Rightarrow \sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} f(\delta \phi \dot{\gamma}^{-1/(\Delta+\beta)}, 1) \]

plot \( \frac{\sigma}{\dot{\gamma}^{\Delta/(\Delta+\beta)}} \) vs \( \frac{\delta \phi}{\dot{\gamma}^{1/(\Delta+\beta)}} \)
model granular material  

bidisperse mixture of soft disks in \textit{two dimensions} at $T = 0$

equal numbers of disks with diameters $d_1 = 1, d_2 = 1.4$

frictionless, harmonic repulsive soft cores

\begin{align*}
V(r) & \propto \delta^2 \\
\delta & \text{ is particle overlap}
\end{align*}

Durian’s overdamped foam dynamics

\[
\frac{d \mathbf{r}_i}{dt} = -D \sum_j \frac{dV(r_{ij})}{d\mathbf{r}_i} + y_i \gamma \hat{\mathbf{x}}
\]

pressure tensor

\[
P_{\alpha\beta} = \frac{1}{L^2} \sum_{i > j} \frac{r_{ij} \alpha r_{ij} \beta}{r_{ij}} \frac{dV}{d\mathbf{r}_{ij}}
\]

Lees-Edwards boundary Conditions for shear flow

shear stress \hspace{1cm} \sigma = p_{xy}

pressure \hspace{1cm} p = \frac{1}{2} (p_{xx} + p_{yy})

energy \hspace{1cm} u = \frac{1}{L^2} \sum_{i > j} V(r_{ij})
simulation parameters

present work

fixed $\dot{\gamma}$, $\sigma(t)$ fluctuates
$N = 65,536$

scaling collapse data to

$$\sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} f\left(\delta \phi \cdot \dot{\gamma}^{-1/(\Delta+\beta)}\right)$$

expand scaling function in polynomial for small values of its argument

$$\sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} \sum_{n} c_n \left[(\phi - \phi_J) \cdot \dot{\gamma}^{-1/(\Delta+\beta)}\right]^n$$

determine $\phi_J$, $\Delta$, $\beta$, $c_n$ from best fit of data to this form

assumes finite size effects negligible, $\xi << L$ (can’t get too close to $\phi_J$)
simulations with finite shear strain rate

\[ \dot{\gamma} = 2 \times 10^{-5} \]

\[ \dot{\gamma} = 10^{-8} \]

\[ \gamma \times 10^8 \]

\[ \phi_f = 0.8431 \]
\[ \Delta = 1.17 \]
\[ \beta = 2.14 \]

\[ (\phi - \phi_f) \dot{\gamma}^{-1/\Delta} \beta \]

\[ N = 65,536 \]
quasi-static simulations $\gamma \to 0$

when the strain rate is sufficiently slow, system will always relax to be in an instantaneous local minimum of the interaction energy

Dynamics: increase strain in fixed steps $\Delta \gamma$, at each step use conjugate gradient method to relax the particle positions to local energy minimum

Maloney & Lemaitre, PRE 2006, $\phi > \phi_J$

Heussinger & Barrat, cond-mat 2009, at point J

$\gamma_{\text{max}} = 50$ to $150$ depending on system size
FIG. 2. Stress time series for several $\dot{\gamma}$ (mHz) for the smaller $R_f$.

Behringer, Bi, Chakraborty, Henkes, Hartley, PRL 101, 268301 (2008)
finite size scaling

\[ L \equiv \sqrt{N} \]

\[ \sigma = b^{-\Delta/\nu} f(\delta \phi b^{1/\nu}, \dot{\gamma} b^z) L b^{-1} \]

add system size \( L \) as new scaling parameter

choose \( b = L, \dot{\gamma} = 0 \) for quasi-static limit

\[ \sigma = L^{-\Delta/\nu} f(\delta \phi L^{1/\nu}, 0, 1) \]

exactly at \( \phi_j \)

\[ \sigma \sim L^{-\Delta/\nu} \]

Hatano, cond-mat 2008

plot \( \sigma L^{\Delta/\nu} \) vs \( \delta \phi L^{1/\nu} \)

data collapse determines exponents \( \Delta \) and \( \nu \)

fit data to polynomial expansion of scaling function

bonus of finite size scaling is that it gives the correlation length exponent \( \nu \) without having to explicitly compute the correlation length!
\[ \sigma = L^{-\Delta/\nu} f(\delta\phi, L^{1/\nu}, 0, 1) \]
energy density $u$, pressure $p$

\[ u \sim \delta \phi^{y_u} \Rightarrow u = L^{-y_u/\nu} U(\delta \phi L^{1/\nu}) \]

\[ p \sim \delta \phi^{y_p} \Rightarrow p = L^{-y_p/\nu} P(\delta \phi L^{1/\nu}) \]

\[ p = \phi \frac{du}{d\phi} \Rightarrow y_p = y_u - 1 \]


effective temperature: Daniel Valdez-Balderas, P14-9, Wed 9:36 am
## Results

Finite Size Scaling with $N = 64$ to $1024$

<table>
<thead>
<tr>
<th></th>
<th>Shear Stress</th>
<th>Energy Density</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_J$</td>
<td>$0.08429 \pm 0.0001$</td>
<td>$0.08425 \pm 0.0001$</td>
<td>$0.08427 \pm 0.0001$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.70 \pm 0.05$</td>
<td>$0.73 \pm 0.04$</td>
<td>$0.70 \pm 0.05$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$1.2 \pm 0.1$</td>
<td>$2.5 \pm 0.1$</td>
<td>$1.3 \pm 0.1$</td>
</tr>
</tbody>
</table>

Check if in scaling limit: Finite Size Scaling with $N = 128$ to $1024$

<table>
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<tr>
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<th>Shear Stress</th>
<th>Energy Density</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_J$</td>
<td>$0.08431 \pm 0.0001$</td>
<td>$0.08428 \pm 0.0001$</td>
<td>$0.08429 \pm 0.0001$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.78 \pm 0.07$</td>
<td>$0.80 \pm 0.05$</td>
<td>$0.79 \pm 0.07$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$1.2 \pm 0.1$</td>
<td>$2.5 \pm 0.2$</td>
<td>$1.3 \pm 0.1$</td>
</tr>
</tbody>
</table>

$\phi_J$, $\nu$ increase  \quad  $\Delta$, $y_u$, $y_p$ stay the same

Wyart, Nagel, Witten, Europhys Lett 2005:  $\nu = 1/2$  soft modes of jammed solid

O’Hern, et al., PRE 2003 + Drocco, et al., PRL 2005:  $\nu \sim 0.7$  numerical

Mike Moore (private communication):  $\nu \sim 0.78$  mapping to spin glass
exponents and ensembles

we find

\[ \Delta = 1.2 \pm 0.1, \quad y_u = 2.5 \pm 0.3, \quad y_p = 1.3 \pm 0.1 \]

“classical values” for harmonic soft cores

\[ \Delta = 1, \quad y_u = 2, \quad y_p = 1 \]

O’Hern, et al: \( y_p = 1 \)

for harmonic soft cores

Why the difference?

*Jammed ensemble depends on the preparation protocol*

fraction of jammed states

Peter Olsson
J14-10

O’Hern et al. random protocol \( \equiv \) fast shearing

as shear rate decreases, system explores longer time scales, jamming density increases!
conclusions

quasi-static method looks promising!
need to study larger system sizes $N$
no definitive value for $\phi_J$ or $\nu$ yet, but we are getting there!
preparation protocol can make a difference