

Quasistatic approximation to the scattering of elastic waves by a circular crack^{a)}

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The elastic-wave scattering by a flat crack can be represented by an integral expression involving displacement and strain on the surface of the crack. We have explored the use of a modified static solution as an approximation to the displacement field of a penny-shape crack in the long-wavelength regime and then studied numerically how well it connects with the low-frequency limit of the diffraction regime. Comparisons between this and several other existing approximations are made. We conclude that this quasistatic approximation is useful practically in both the long-wavelength and beginning diffraction regime.

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I. INTRODUCTION

The calculation of the scattering of ultrasonic waves is a fundamental problem in the nondestructive detection of cracks in elastic materials. Using the idealization that a crack is a region of fixed size and shape on which certain boundary conditions for the displacement field are specified, various approximate solutions in both the high- and low-frequency limits can be obtained.

This paper will present a review of some of the existing approximations for the scattering of a normally incident longitudinal wave off a circular crack and then present a new quasistatic approximation for the long-wavelength limit. This new approximation is the only one we know of which clearly gives rise to the long-wavelength Rayleigh limit for the scattered power. As current transducer frequencies are such that for cracks with diameters on the order of 100 μ probing is limited to the long-wavelength region, we feel this approximation is a significant step in the nondestructive analysis program.

There exists considerable literature on the problem of scattering of elastic waves by a circular crack; most of these have been summarized by Kraut.¹ Recently, the formal theory of scattering has been examined from an integral equation viewpoint to obtain concise asymptotic quantities which relate directly to experiment.²

In addition to the new approximation we report in the present study, we considered those given by Filipczynski,³ a modified Kirchhoff approximation, the "half-space" approximation of Miller and Pursey,⁴ one proposed by Mal,^{1,5} and the Keller theory for shear-free media.⁶

In Sec. II, we reduce the basic integral formula for scattering of elastic waves to a form useful for cracks. We then examine the form it takes for

several of the above referenced approximations in Sec. III.

In Sec. IV, we discuss the low-frequency limit (quasistatic) and the results when exact static-elastic solutions for the crack strain field are introduced as leading approximations. An important result is that the correct long-wavelength Rayleigh limit is obtained.

Computed results are compared for the various approximations.

We conclude that the quasistatic approximation is a realistic formulation for computing the scattering from a crack, in a form useful for experimental application.

II. SURFACE-INTEGRAL REPRESENTATIONS

The problem of elastic waves scattered by a void in a homogenous infinite medium is defined as follows.^{1,2} The displacement field $u_i(\mathbf{r})$ satisfies the wave equation

$$C_{ijkl}u_{k,jl} = \rho \ddot{u}_i \quad (2.1)$$

(standard notation for space and time derivations is used; C_{ijkl} is the elastic tensor and ρ is the density of the medium). For harmonic time dependence, Eq. (2.1) reduces to

$$C_{ijkl}u_{k,jl} + \rho\omega^2 u_i = 0. \quad (2.2)$$

Consider the situation of Fig. 1, with V the volume enclosed by a surface $\Sigma = S + S'$, where S is the surface of the defect and S' is a circumscribing surface generally taken to be at infinity. To characterize a scattering problem, we have to specify certain

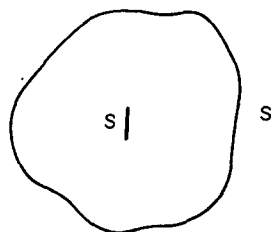


FIG. 1. Surface of integration used in Kirchhoff and quasi-static approximations.

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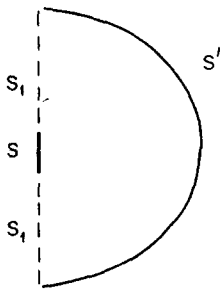


FIG. 2. Surface of integration used in Miller and Pursey approximations.

boundary conditions on S and S' . Since we are interested in a scattering solution with a given incident wave u^0 , we impose

$$u_i(\mathbf{r}) \rightarrow u_i^0(\mathbf{r}) \quad \text{for } \mathbf{r} \in S', \text{ as } S' \rightarrow \infty. \quad (2.3a)$$

The nature of our scatterer imposes the boundary condition on S . For a weak (disbonded) crack, we require

$$\sigma_{ij} n_j = C_{ijkl} n_j u_{k,l} = 0 \quad \text{for } \mathbf{r} \in S, \quad (2.3b)$$

where n_j is the outward normal to the surface of the crack. This condition means that all components of the force on the surface S vanish.

Equation (2.2) with boundary conditions (2.3a) and (2.3b) uniquely determines the scattering solution.

Throughout this work, we shall use a convenient surface-integral representation of the scattering equation. Represent

$$u_i(\mathbf{r}) = u_i^0(\mathbf{r}) + u_i^s(\mathbf{r}); \quad (2.4)$$

then, for any point $\mathbf{r} \in V$, where V is bounded by a surface Σ , we can write

$$u_m^s(\mathbf{r}) = C_{ijkl} \int_{\Sigma} d\Sigma' n_j [g_{im}(\mathbf{r} - \mathbf{r}') u_{k,\nu}^s(\mathbf{r}') - g_{km,\nu}(\mathbf{r} - \mathbf{r}') u_j^s(\mathbf{r}')]. \quad (2.5)$$

Note that the surface Σ need not be a connected surface (see Fig. 1). Here, $u_{k,\nu}^s(\mathbf{r}') = \partial u_k^s(\mathbf{r}') / \partial r'_\nu$, and $g_{im}(\mathbf{r} - \mathbf{r}')$ is the Green's function that satisfies the equation

$$C_{ijkl} g_{km,jl} + \rho \omega^2 g_{im} = -\delta_{im} \delta(\mathbf{r} - \mathbf{r}'). \quad (2.6)$$

Equation (2.5) is the starting point for various approximate treatments.

To specify Eq. (2.5), we need the solution for Eq. (2.6), i. e., the Green's function $g_{ij}(\mathbf{r} - \mathbf{r}')$. To give the correct dependence for the asymptotic displacement field, we must choose a solution of Eq. (2.6) which generates outgoing scattered waves, i. e., for $\exp(-i\omega t)$ time dependence, $g \sim \exp(ikR)$. Other than this requirement, any solution to Eq. (2.6), and, in particular, the infinite medium Green's function, is adequate for use in Eq. (2.5). Alternatively, depending on the particular geometry of the surface Σ , it may be possible to find a Green's function satisfying such boundary conditions on Σ that will eliminate terms in the integral (2.5) and, thus, simplify the problem.

Two types of surfaces will be considered, that of Figs. 1 and 2. For both, we want no contribution

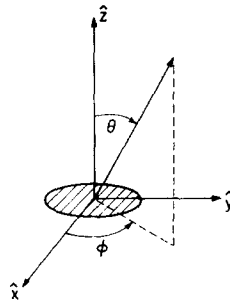


FIG. 3. Geometry of scattering. Incident wave is along \hat{z} , θ , and ϕ specify the point of observation of the scattered field.

from S' to u^s as $S' \rightarrow \infty$, so we impose the additional condition that $g_{ij}(R) \sim \exp(ikR)/R$ as $R \rightarrow \infty$.

One final comment on Eq. (2.5) should be made. While the volume formulation yields an integral equation for the (total field) solution of the scattering problem, since Eq. (2.5) nowhere involves the incident wave u^0 , it is not an integral equation, but rather a representation of u^s on Σ .

III. REVIEW AND COMPARISON OF EXISTING APPROXIMATIONS

To explore and compare various approximations here, we have considered only normally incident longitudinal waves. The scattered wave is given, in the far-field limit, by²

$$u^s = \hat{r}A[\exp(i\alpha r)/r] + \hat{\theta}B[\exp(i\beta r)/r]. \quad (3.1)$$

Here, α and β are the longitudinal- and transverse-wave vectors [$\beta = (v_L/v_T)\alpha$, where v_L and v_T are the longitudinal- and transverse-wave velocities]. Symmetry requires the transverse part to be solely in the θ direction. The results of various approximations specify the dependence of A and B on the scattering angle θ , the wave vector of the incident wave α , and the radius of the crack, a . The scattering angle θ and the geometry of the scattering situation are shown in Fig. 3.

The scattered power into solid angle $d\Omega$ is given by

$$\frac{dP}{d\Omega} = \frac{dP_L}{d\Omega} + \frac{\alpha}{\beta} \frac{dP_T}{d\Omega}, \quad (3.2)$$

$$\frac{dP_L}{d\Omega} = |A|^2, \quad \frac{dP_T}{d\Omega} = |B|^2. \quad (3.3)$$

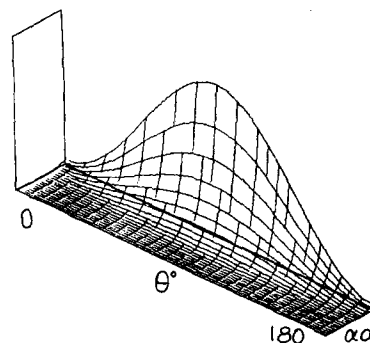


FIG. 4. Total scattered power in Filipczynski approximation, $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

We now quote the results of various approximations and present plots of scattered power as a function of αa and θ .

(a) Filipczynski³ approached the problem by a separation of variables in an axially symmetric oblate spheroidal coordinate system for the long-wavelength limit. His results are

$$A(\alpha, \theta) = -i\alpha \varphi_A \cos\theta, \quad (3.4a)$$

$$B(\alpha, \theta) = i\beta \varphi_B \sin\theta, \quad (3.4b)$$

where

$$\varphi_A = -\frac{2a}{3\pi} \frac{(\alpha a)^2}{1 + \frac{3}{4}(\beta/\alpha)^2}, \quad \varphi_B = -\frac{2a}{3\pi} \frac{(\beta a)^2}{1 + \frac{3}{4}(\beta/\alpha)^2}. \quad (3.5)$$

Note that for low α , the scattered power $\sim \alpha^6$ (Figs. 4–6).

(b) Modified Kirchhoff. This approximation uses Eq. (2.5) with Σ of Fig. 1 as the starting point. Denote the illuminated side of the crack by S^- and the other side by S^+ . The contribution of the crack can be written in terms of jumps across S :

$$u_m^s(\mathbf{r}) = C_{ijkl} \int_S ds' n_j^* \{ g_{im}(\mathbf{r} - \mathbf{r}') [u_{k,r}^s] - g_{km,r}(\mathbf{r} - \mathbf{r}') [u_i^s] \}, \quad (3.6)$$

where $[u^s]$ denotes the jump;

$$[u^s] = u^s(\mathbf{r}' \in S^+) - u^s(\mathbf{r}' \in S^-). \quad (3.7)$$

Since the jump in the incident field u^0 is trivially zero, the “weak crack” boundary condition imposes

$$C_{ijkl} n_j [u_{k,l}^s] = 0, \quad (3.8)$$

and, thus, Eq. (3.6) reduces to

$$u_m^s(\mathbf{r}) = -C_{ijkl} \int_S ds' n_j^* g_{km,r}(\mathbf{r} - \mathbf{r}') [u_i^s], \quad (3.9)$$

where $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ for a homogeneous isotropic medium.

For a circular crack and longitudinal wave incident along the axis of symmetry, the integration over the angular variable can be done (we assume that under these circumstances only $[u_z^s] \neq 0$ and that this jump depends only on the radial variable r). We obtain from Eq. (3.9), in the far-field limit,

$$|A|^2 = \frac{\alpha^2}{4} \left(\frac{\lambda + 2\mu \cos^2\theta}{\lambda + 2\mu} \right)^2 |I(\alpha)|^2, \quad (3.10)$$

$$|B|^2 = \beta^2 \cos^2\theta \sin^2\theta |I(\beta)|^2, \quad (3.11)$$

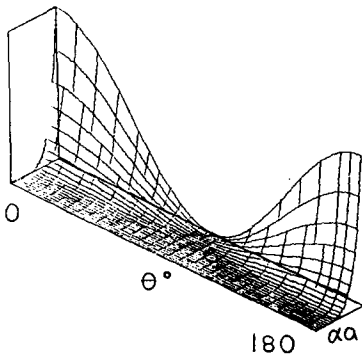


FIG. 5. Longitudinal scattered power in Filipczynski approximation. $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

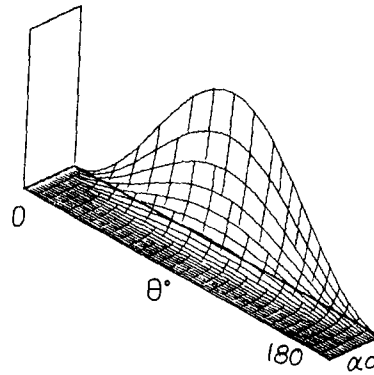


FIG. 6. Transverse scattered power in Filipczynski approximation. $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

where

$$I(\gamma) = \int_0^a r dr \Delta u(r) J_0(\gamma r \sin\theta) \quad (3.12)$$

and $\Delta u(r) = [u_z^s]$ is the jump (in the z component) of the displacement field over the crack.

Our modified Kirchhoff approximation now consists of making the following guess for $\Delta u(r)$. On the back side of the crack S^+ , we take $u = 0$. On the illuminated side S^- , we postulate a reflected wave u_R which will give a zero total stress on S^- . For a plane incident wave in the positive z direction, the simplest choice is one giving a standing wave,

$$u_R = \exp(-ikz) \hat{z}, \quad (3.13)$$

where $z = 0$ is the plane of the crack.

The total displacement is then

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + \mathbf{u}_R \\ &= [\exp(ikz) + \exp(-ikz)] \hat{z} \\ &= 2 \cos(kz) \hat{z}, \end{aligned} \quad (3.14)$$

and the strain is

$$\begin{aligned} u_{i,j} &= -2k \sin kz \delta_{i,3} \delta_{j,3} \\ &= 0 \end{aligned} \quad (3.15)$$

on the crack surface S^- .

This guess corresponds to a ray picture with the wave fronts remaining parallel to the surface of the crack. As in geometric optics, this approximation is expected to be good only in the high-frequency limit.

Using these assumptions, one gets for the jump in displacement across the crack,

$$\Delta u(r) = 2, \quad 0 \leq r \leq a \quad (3.16)$$

which when used in Eq. (3.12) gives the results

$$|A|^2 = \alpha^2 a^4 \left(\frac{\lambda + 2\mu \cos^2\theta}{\lambda + 2\mu} \right)^2 \left(\frac{J_1(\alpha a \sin\theta)}{\alpha a \sin\theta} \right)^2, \quad (3.17)$$

$$|B|^2 = 4\beta^2 a^4 \cos^2\theta \sin^2\theta \left(\frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta} \right)^2, \quad (3.18)$$

where $J_1(x)$ is the Bessel function of order unity and

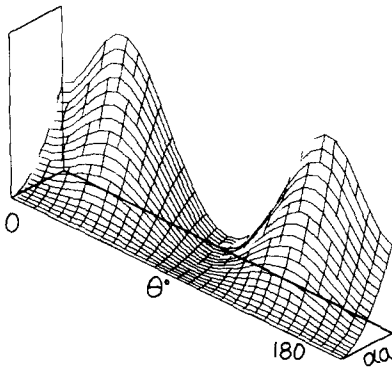


FIG. 7. Total scattered power in modified Kirchhoff approximation. $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

for small x

$$J_1(x) \sim \frac{1}{2}x. \quad (3.19)$$

Thus, we have for small α ,

$$\frac{dP}{d\Omega} \sim \alpha^2. \quad (3.20)$$

The modified Kirchhoff approximation presented here differs from the standard Kirchhoff approximation⁷ by use of the exact boundary condition (3.8) and the postulation of a reflected wave on the illuminated side of the crack. The standard Kirchhoff approximation neglects these and assumes the jump in displacement and strain across the crack to be equal to the displacement and strain of the incident field at the surface of the crack, thus modeling scattering from a perfect absorber (Figs. 7–9).

(c) Keller⁷ has developed an extension of geometric optics to include diffraction. His approximation for the acoustic case (i. e., shear-free medium) gives the amplitude as

$$A(\theta) = \frac{a}{2} \left(\frac{iJ_1(\alpha a \sin\theta)}{\sin(\theta/2)} - \frac{J_0(\alpha a \sin\theta)}{\cos(\theta/2)} \right). \quad (3.21)$$

This result was compared to the modified Kirchhoff approximation in a shearless medium for an αa of 15. The results are virtually identical and are given in Fig. 10.

(d) Miller and Pursey treated the problem of the

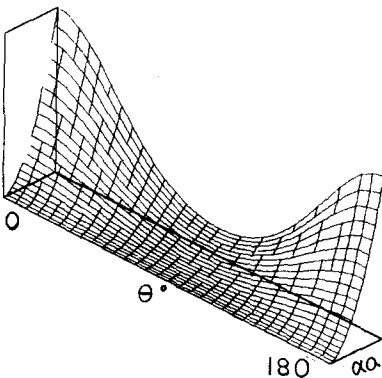


FIG. 8. Longitudinal scattered power in modified Kirchhoff approximation. $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

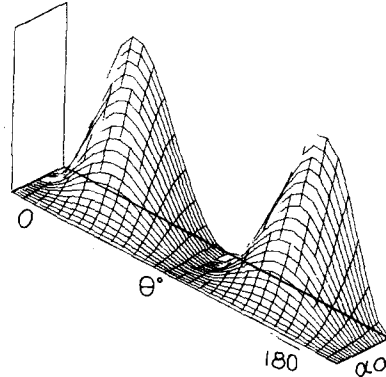


FIG. 9. Transverse scattered power in modified Kirchhoff approximation. $\alpha a = 0$ to 1, $\theta = 0^\circ$ to 180° .

field due to an oscillating normal stress applied over a circular region of radius a on an otherwise free surface of a semi-infinite elastic medium. The approximate boundary conditions on the $z = 0$ plane are

$$\begin{aligned} C_{33ij}u_{i,j} &= 1 \quad \text{for } r \leq a, \\ C_{33ij}u_{i,j} &= 0 \quad \text{for } r > a, \\ C_{13ij}u_{i,j} &= C_{23ij}u_{i,j} \\ &= 0 \quad \text{for } 0 \leq r < \infty. \end{aligned}$$

This problem relates to our scattering problem as follows. Using Eq. (2.5) with Σ as in Fig. 2, if we can find⁸ a Green's function \tilde{g} such that $\tilde{g}_{km,i}(z=0) = 0$, then Eq. (2.5) reduces to

$$u_m^s(r) = C_{ijkl} \int_{S+S_1} ds' n_j \tilde{g}_{im} u_{k,l}^s, \quad (3.22)$$

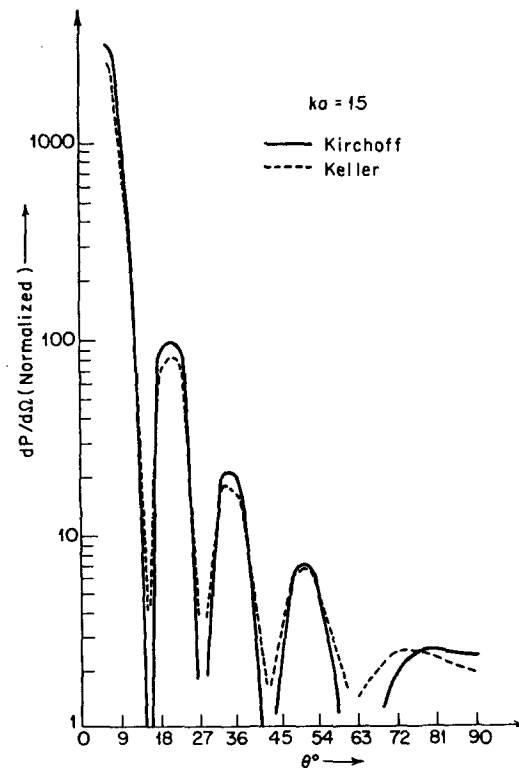


FIG. 10. Kirchhoff versus Keller approximation for $\alpha a = 15$, $\theta = 0^\circ$ to 90° in a shearless medium.

where S is the surface of the crack, and S_1 is the remainder of the x - y plane. Assuming now (for $z=0^+$)

$$\begin{aligned} u_{k,l}^s &= -u_{k,l}^0 \quad \text{on } S \\ &= 0 \quad \text{on } S_1, \end{aligned} \quad (3.23)$$

we get the Miller-Purse solution for forward scattering (extended, by symmetry, to $90^\circ < \theta < 180^\circ$), whose form for low αa is given in Kraut's review article. The scattered power varies as α^2 (Fig. 11).

IV. QUASISTATIC APPROXIMATION (Ref. 9)

We return now to Eqs. (3.10) to (3.12). The modified Kirchhoff approximation consisted of making some guess for $\Delta u(r)$ in Eq. (3.12) that would be reasonable for high frequencies.

The quasistatic approximation now consists of making what we believe to be a good guess for $\Delta u(r)$ in the long-wavelength limit. Before making this approximation, we note that certain general conclusions can be reached from Eqs. (3.10)–(3.12) without specifying $\Delta u(r)$. (a) The scattered power is symmetric about the $z=0$ plane. (b) The transverse scattered power vanishes at $\theta=0^\circ$, 90° , and 180° .

For small αa and $\beta/\alpha \approx 2$, it is reasonable to assume that $\text{Re}[\Delta u(r)]$ and $\text{Im}[\Delta u(r)]$ do not change sign in $0 < r < a$. Also note that $J_0(x)$ is a positive and decreasing function for $0 < x < \alpha a$. Then we have the following: (c) The transverse scattered power has absolute maximum B_{max}^2 at $\theta \approx 45^\circ$ and 135° . (d) The longitudinal scattered power will have absolute maximum value A_{max}^2 at $\theta=0^\circ$ and 180° and a nonzero absolute minimum A_{min}^2 at $\theta=90^\circ$. (e) Order-of-magnitude estimates of various scattered power ratios are

$$\begin{aligned} A_{\text{max}}^2/B_{\text{max}}^2 &= O(0.1), \\ A_{\text{max}}^2/A_{\text{min}}^2 &= O(1) - O(10). \end{aligned}$$

These general features are exhibited by Mal's results⁵ as contained in Fig. 5 of Kraut.

To obtain a good guess for $\Delta u(r)$, we consider the static problem defined by (note: $\sigma_{ij} = C_{ijkl} u_{k,l}$)

$$\sigma_{ij,j} = 0 \quad (4.1)$$

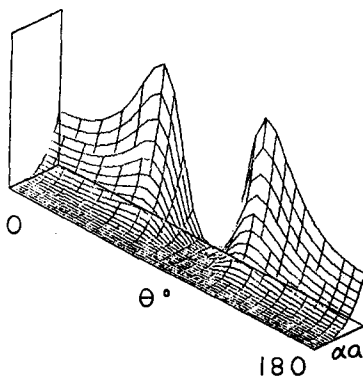


FIG. 11. Longitudinal scattered power in Miller-Purse approximation.

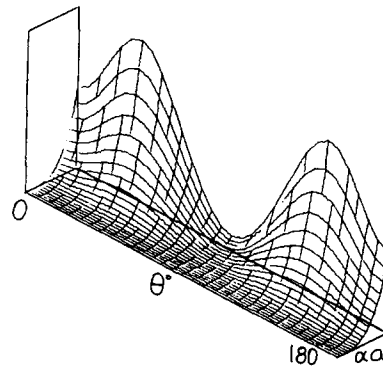


FIG. 12. Total scattered power in quasistatic approximation, $\alpha a=0$ to 1, $\theta=0^\circ$ to 180° .

with boundary conditions

$$\begin{aligned} \sigma_{ij} &= \sigma_{ij}^0 \quad \text{at } z = \pm \infty, \\ \sigma_{3j} &= 0 \quad \text{on crack.} \end{aligned} \quad (4.2)$$

Defining σ_{ij}^s by

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^s, \quad (4.3)$$

then the static problem for σ_{ij}^s takes the form

$$\sigma_{ij,j}^s = 0 \quad (4.4)$$

with boundary conditions

$$\begin{aligned} \sigma_{ij}^s &= 0 \quad \text{at } z = \pm \infty \\ \sigma_{3j}^s &= -\sigma_{3j}^0 \quad \text{on crack.} \end{aligned} \quad (4.5)$$

This problem has been solved¹⁰; the solution yields $\Delta u(r)$ of the form

$$\Delta u(r) = Q \sigma_{33}^0 a (1 - r^2/a^2)^{1/2}, \quad r \leq a, \quad (4.6)$$

where Q is a constant.¹⁰ For our quasistatic approximation we use the long-wavelength limit of the incident stress field as our σ^0 ; for a longitudinal plane wave, this is

$$\sigma_{33}^0 \cong i\alpha(\lambda + 2\mu),$$

and we get

$$\Delta u(r) = Q' a \alpha (1 - r^2/a^2)^{1/2}. \quad (4.7)$$

Dropping the dimensionless constant Q' , we get, upon substitution into Eq. (3.12),

$$\begin{aligned} I(\gamma) &= \alpha a \int_0^a r dr (1 - r^2/a^2)^{1/2} J_0(\gamma r \sin\theta) \\ &= \alpha a \frac{1}{\gamma^2 \sin^2\theta} \left[\frac{\sin(\gamma a \sin\theta)}{\gamma a \sin\theta} - \cos(\gamma a \sin\theta) \right] \end{aligned}$$

Substitution into Eqs. (3.10) and (3.11) yields the scattered power which for low α goes as

$$\frac{dP}{d\Omega} \sim \alpha^4$$

and agrees with the long-wavelength Rayleigh limit (Figs. 12–14).

V. SUMMARY

The problem of scattering of elastic waves off a

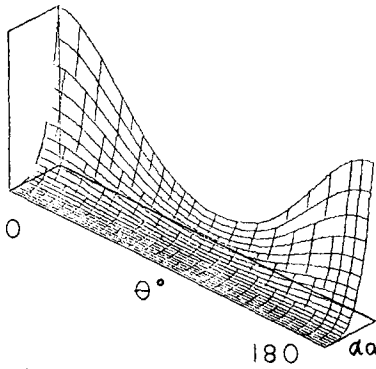


FIG. 13. Longitudinal scattered power in quasistatic approximation. $\alpha a=0$ to 1, $\theta=0^\circ$ to 180° .

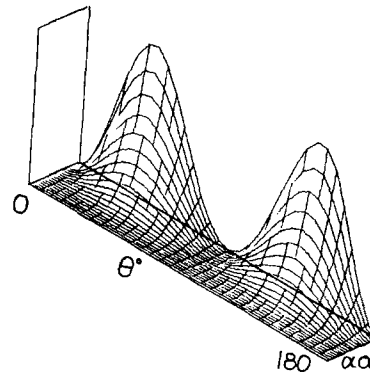


FIG. 14. Transverse scattered power in quasistatic approximation. $\alpha a=0$ to 1, $\theta=0^\circ$ to 180° .

circular crack has been studied extensively. The modified Kirchhoff approximation is expected to be good in the high-frequency limit. The Keller theory is also expected to be good in this region, but has yet to be extended to a medium with nonzero shear. The results, however, in the low-frequency region are less satisfactory. The Filipczynski approximation appears inconsistent with the general results of Sec. IV in the shape of the transverse power distribution and in the relative magnitudes of transverse and longitudinal power. The Miller-Pursey approximation starts off by assuming Kirchhoff-like (and, hence, high-frequency-like) boundary conditions, but yields a convenient analytic result only in the low-frequency limit. It also appears inconsistent with the general results of Sec. IV, concerning the shape of the longitudinal power distribution. The results of Mal agree with the general results of Sec. IV; however, they are expressed in terms of numerically calculated amplitude functions which are less convenient to deal with than an analytic expression.

Since the low-frequency region is of current experimental interest, it is important to put the theory in this region on firm ground. We feel that the quasistatic approximation achieves this goal. By using the integral representation formulation and the physically reasonable assertion that the opening and closing of the crack $\Delta u(r)$ cannot vary rapidly with r , we achieve general criteria for solutions to the problem. These criteria place strict limits on the form of the solution and are very useful in comparing contending approximations. By plugging in the most natural guess for $\Delta u(r)$, e.g., the static solution, we arrive at a simple analytic expression for the scattered power. This solution trivially meets all the aforementioned conditions. It also clearly obeys the frequency dependency of the long-wavelength Rayleigh limit (ω^4) for the scattered power. Comparing the angular dependence of the scattered power for the quasistatic and modified Kirchhoff approximations (which are expected to be good for entirely different frequency regions,) the only major difference appears in the overall frequency dependence. While Eqs. (3.10)–(3.12) show that this must be the case for αa close to zero, it is surprising how well they continue

to agree up to an αa of unity. Thus, the quasistatic approximation maps nicely onto the modified Kirchhoff approximation and should prove useful over a fairly large range of frequencies.

The results presented are in simple analytic form and, thus, should be useful and convenient for comparison with experimental data and suggesting what experiments will best test the theory.

Since the static solution for a pure shear stress on a circular crack is also known, this solution could be used in the integral representation presented to model a normally incident transverse wave. Combining the pure static shear solution with the pure static compressional solution yields approximations for arbitrary angle of incidence of transverse or longitudinal waves. Generalization to elliptically shaped cracks should also be straightforward.

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¹E. A. Kraut, IEEE Trans. Sonics Ultrason. SU-23, 162 (1976).

²J. E. Gubernatis, E. Domany, and J. A. Krumhansl, J. Appl. Phys. 48, 2804 (1977).

³L. Filipczynski, in Proc. of Vibration Problems, Warsaw, Poland, 1961, Vol. 2 (unpublished).

⁴G. F. Miller and H. Pursey, Proc. R. Soc. London A 223, 521 (1954).

⁵A. K. Mal, Int. J. Eng. Sci. 8, 381 (1970).

⁶J. B. Keller, J. Opt. Soc. Am. 52, 116 (1962).

⁷See discussion in Ref. 1.

⁸H. Lamb, Philos. Trans. R. Soc. London A 203, 1 (1904).

⁹The utilization of various static solutions as the zeroth-order approximation in the integral equation for volume scatterers was first proposed by J. E. Gubernatis. The approach taken here for scattering by cracks was suggested independently by J. E. Gubernatis, R. Thompson, and J. A. Krumhansl, [J. Appl. Phys. (to be published)].

¹⁰I. N. Sneddon, *Fourier Transforms* (McGraw-Hill, New York, 1951), p. 490.