Determination of crack characteristics from the quasistatic approximation for the scattering of elastic waves

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We extend the work of a previous paper to give a long-wavelength approximation for elastic wave scattering by an elliptical flat crack. Explicit formulas for the far-field scattered amplitudes in this approximation are given for various experimental configurations. These formulas are then applied to give a simple inversion procedure. The orientation and eccentricity of the crack are readily determined from unnormalized scattering data. The size may be determined from an absolute intensity measurement.

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I. INTRODUCTION

The scattering of ultrasonic waves off cracks in elastic materials and the inversion of such scattering data to yield information as to the nature of the crack is an important technique for the nondestructive testing of elastic materials.

Current transducer frequencies are such that for cracks with a diameter on the order of $100\,\mu$ probing is limited to the long-wavelength region and direct imaging of the scatterer is not possible. In this paper, we therefore consider the problem of long-wavelength scattering off an elliptical crack and present an inversion scheme whereby all features of the elliptical crack may be determined from the long-wavelength scattering data.

In a previous paper,[1] henceforth referred to as I, we reviewed the problem of scattering of a normally incident longitudinal wave by a circular crack and presented a new quasistatic approximation found to be good in the long-wavelength region. In Sec. II of this paper, we formally extend this approximation to the case of longitudinal and transverse waves at an arbitrary angle of incidence scattered by an elliptical crack.

Explicit formulas are presented for the far-field scattered amplitudes in this approximation for several cases of experimental interest. In Sec. III, we apply the results of Sec. II to the particular case of backscattering from a longitudinal incident wave and show how this scattering data can be inverted in a simple fashion to determine all features of the elliptical crack. The orientation and eccentricity of the crack are obtainable from unnormalized scattering data. Determination of the size requires an absolute intensity measurement.

We conclude that even in the long-wavelength limit it is straightforward to determine all information as to the nature of the elliptical crack.

II. QUASISTATIC APPROXIMATION FOR ELLIPTICAL CRACKS

The integral representation formalism for scattering as applied to cracks and the basic motivation and justification for the quasistatic approximation has been presented in I. Proceeding as in I, we model the crack as a stress-free surface of fixed shape. One can express the far-field scattered amplitudes in terms of the vector[2]

$$f_i(\mathbf{k})=\frac{ik^3}{4\pi\rho\omega^2}C_{ijkl}\hat{r}_j\int_{S^+}ds'\,n_k^+[u_l]\exp(-i\mathbf{k}\cdot\mathbf{r}'), \tag{1}$$

where n_k^+ is the outward normal of the side S^+ of the crack, ω is the frequency of the incident wave, and C_{ijkl} and ρ are the elastic tensor and density of the medium. $[u_l]$ is the jump in the lth component of the displacement field across the surface

$$[u_l]=u_l(r'\epsilon S^+)-u_l(r'\epsilon S^-)\;. \tag{2}$$

In terms of the vector f the far-field amplitude for the longitudinal scattered wave is

$$A_i(\hat{r})=\hat{r}_i\hat{r}_jf_j(\boldsymbol{\alpha}) \tag{3}$$

and for the transverse scattered wave is

$$B_i(\hat{r})=(\delta_{ij}-\hat{r}_i\hat{r}_j)f_j(\boldsymbol{\beta}), \tag{4}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the longitudinal and transverse wave vectors corresponding to ω and point from the center of the crack to the point of observation.

For the case of an elliptical crack lying in the xy plane with major axis along x in a homogeneous isotropic medium.

$$\begin{aligned}C_{ijkl}&=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}),\\ n_k^+&=\delta_{3k}.\end{aligned} \tag{5}$$

In the static limit $[u_l]$ is proportional to T_{l3} where T_{ij} is the stress tensor on the surface of the crack in the coordinate system specified above. The exact solution for $[u_l]$ in the static case can be obtained from the work of Eshelby[3]

$$\begin{aligned}[][u_1]&=c_1T_{13}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2},\\ [u_2]&=c_2T_{23}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2},\\ [u_3]&=c_3T_{33}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2},\end{aligned} \tag{6}$$

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where for the ellipse bounded by $(x^2/a^2)+(y^2/b^2)=1$ and $a>b$ we have

$$c_3=\frac{4b}{E(\kappa)}\frac{1-\sigma^2}{Y},\quad c_2=\frac{2a}{\mu\eta_2},\quad c_1=\frac{2b}{\mu\eta_1},\tag{7}$$

where

$$\eta_1=E(\kappa)+\frac{\sigma}{1-\sigma}\frac{\kappa'^2}{\kappa^2}[K(\kappa)-E(\kappa)],$$

$$\eta_2=\frac{E(\kappa)}{\kappa'}+\frac{\sigma}{1-\sigma}\frac{1}{\kappa'}\frac{E(\kappa)-\kappa'^2K(\kappa)}{\kappa^2},$$

$$\kappa=(1-b^2/a^2)^{1/2},\ \kappa'=b/a.$$

For $a=b$,

$$\eta_1=\eta_2=\frac{\pi(2-\sigma)}{4(1-\sigma)}.$$

Here σ is Poisson's ratio, μ is the shear modulus, Y is Young's modulus, and $E(\kappa)$ and $K(\kappa)$ are complete elliptic integrals of second and first kind, respectively, with modulus κ.

For an incident wave $\mathbf{u}_0=\hat{p}\exp(ik_i\mathbf{r}\cdot\hat{d})$ with polarization $\hat{p}$ propagating in direction $\hat{d}$, the stress T_{l3} will have the form

$$T_{l3}=\tau_{l3}ik_i\exp(i\mathbf{k}_i\cdot\mathbf{r}),\tag{8}$$

where

$$\tau_{l3}=C_{l3ij}\hat{p}_i\hat{d}_j.$$

In the quasistatic approximation, we assume that at low k_i the relations given by Eq. (6) are still accurate enough to use in the integral Eq. (1). Substituting Eq. (8) into Eq. (6) and then Eqs. (6) and (5) into Eq. (1) results in

$$f_i(\mathbf{k})=-\frac{k^3k_i}{4\pi\rho\omega^2}I(\mathbf{k}_i-\mathbf{k})\{\mu c_1\tau_{13}(\delta_{i1}\hat{r}_3+\delta_{i3}\hat{r}_1)$$

$$+\mu c_2\tau_{23}(\delta_{i2}\hat{r}_3+\delta_{i3}\hat{r}_2)$$

$$+c_3\tau_{33}(\lambda\hat{r}_i+2\mu\delta_{i3}\hat{r}_3)\},\tag{9}$$

where

$$I(\gamma)=2\pi ab\int_0^1 dr\,r(1-r^2)^{1/2}J_0[\gamma r\sin\theta_\gamma$$

$$\times(a^2\cos^2\varphi_\gamma+b^2\sin^2\varphi_\gamma)^{1/2}]$$

$$=2\pi ab\frac{1}{\Delta^2}\left(\frac{\sin\Delta}{\Delta}-\cos\Delta\right),$$

$$\Delta=(a^2\gamma_x^2+b^2\gamma_y^2)^{1/2}.$$

We now substitute this f vector into Eqs. (3) and (4) to get

$$\mathbf{A}=-\hat{r}\frac{\alpha^3k_i}{4\pi\rho\omega^2}I(\mathbf{k}_i-\boldsymbol{\alpha})\{c_3\tau_{33}(\lambda+2\mu\cos^2\theta)$$

$$+\mu\sin2\theta(c_1\tau_{13}\cos\varphi+c_2\tau_{23}\sin\varphi)\},\tag{10}$$

$$\mathbf{B}=-\frac{\beta^3k_i}{4\pi\rho\omega^2}I(\mathbf{k}_i-\boldsymbol{\beta})\mu$$

$$\times\{\hat{e}_\theta[c_3\tau_{33}\sin2\theta-\cos2\theta(c_1\tau_{13}\cos\varphi+c_2\tau_{23}\sin\varphi)]$$

$$+\hat{e}_\varphi\cos\theta(c_2\tau_{23}\cos\varphi-c_1\tau_{13}\sin\varphi)\}.\tag{11}$$

Here the angles specify the point of observation with respect to the coordinate frame of the crack. Note that the amplitudes are proportional to k_i^2, so that the approximation obeys the long-wavelength Rayleigh limit with power proportional to ω^4.

In order to put these formulas to use we must now calculate τ_{i3} for various experimental configurations.

For an incident wave coming in at polar angles θ_i and φ_i with respect to the coordinate frame of the crack we consider the three polarizations given by the axes of the primed coordinate system shown in Fig. 1. $\hat{z}'$ is the direction of the incident wave, $\hat{x}'$ lies in the plane of the crack and is orthogonal to $\hat{z}'$, and $\hat{y}'$ completes the right-hand coordinate system.

We consider the three cases:

(1) Longitudinal incident wave polarized along $\hat{z}'$.

For an incident displacement field of unit amplitude,

$$\mathbf{u}_0=\hat{z}'\exp(ik_iz')$$

we have

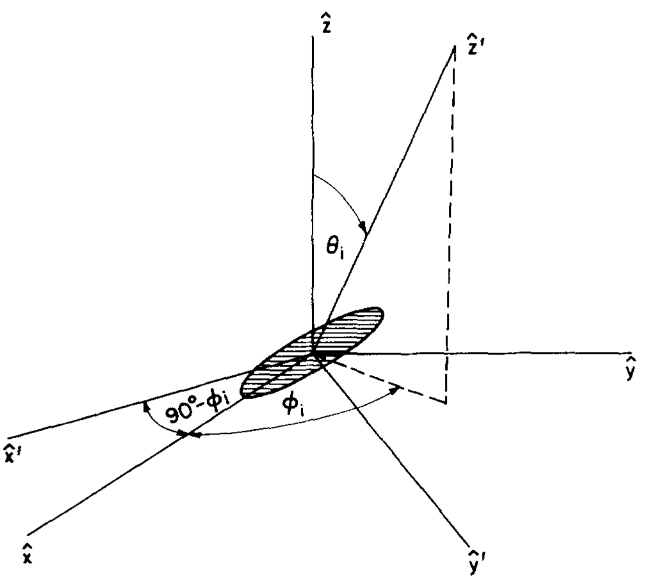

FIG. 1. Geometry of scattering. Incident waves travels along $\hat{z}$, and is polarized along $\hat{x}'$, $\hat{y}'$, or $\hat{z}'$. The crack lies in xy plane with major axis along $\hat{x}$. Scattered field is given as a function of spherical angles with respect to the unprimed coordinate system.

$$\tau_{13}=\mu \sin 2\theta_i \cos\varphi_i,$$
$$\tau_{23}=\mu \sin 2\theta_i \sin\varphi_i, \tag{12}$$
$$\tau_{33}=\lambda+2\mu \cos^2\theta_i.$$

(2) Transverse incident wave polarized in plane of crack along x'. For $\mathbf{u}_0=\hat{x}'\exp(ikz')$ we have

$$\tau_{13}=\mu \cos\theta_i \sin\varphi_i,$$
$$\tau_{23}=-\mu \cos\theta_i \cos\varphi_i, \tag{13}$$
$$\tau_{33}=0.$$

(3) Transverse incident w[ave polarized out of plane of crack along $\hat{y}'$. For $\mathbf{u}_0=\hat{y}'\exp(ikz')$ we have

$$\tau_{13}=\mu \cos 2\theta_i \cos\varphi_i,$$
$$\tau_{23}=\mu \cos 2\theta_i \sin\varphi_i, \tag{14}$$
$$\tau_{33}=-\mu \sin 2\theta_i.$$

In order to compare with the experiment one now chooses the appropriate τ_{i3} listed above. Substituting them into Eqs. (10) and (11) completely determines the far-field amplitudes in the quasistatic approximation.

III. INVERSION OF SCATTERING DATA

Equations (10) and (11) predict scattering amplitudes at all angles for any given incident wave. They contain more information than is needed for the purpose of determining the nature of the elliptical crack. We therefore restrict ourselves to the case of backscattering from a longitudinal incident wave. This corresponds to the experimental case of a movable transducer emitting a longitudinal wave and measuring the longitudinal and transverse waves reflected back (pulse echo mode). Substituting Eqs. (12) into Eqs. (10) and (11) and setting $\theta=\theta_i$ and $\varphi=\varphi_i$ we get,

$$A(\theta,\varphi)=\frac{-\alpha^3 k_i}{4\pi\rho\omega^2} I(\mathbf{k}_i-\boldsymbol{\alpha})\{c_3(\lambda+2\mu\cos^2\theta)^2$$
$$+\mu^2\sin^2 2\theta\,[c_1+(c_2-c_1)\sin^2\varphi]\}, \tag{15}$$

$$B_\theta(\theta,\varphi)=\frac{-\beta^3 k_i}{4\pi\rho\omega^2} I(\mathbf{k}_i-\boldsymbol{\beta})\{c_3\mu\sin 2\theta(\lambda+2\mu\cos^2\theta)$$
$$-\tfrac{1}{2}\mu^2\sin 4\theta\,[c_1+(c_2-c_1)\sin^2\varphi]\}, \tag{16}$$

$$B_\varphi(\theta,\varphi)=\frac{-\beta^3 k_i}{4\pi\rho\omega^2} I(\mathbf{k}_i-\boldsymbol{\beta})\tfrac{1}{2}\mu^2$$
$$\times\cos\theta\sin 2\theta\sin 2\varphi\,(c_2-c_1). \tag{17}$$

Here B_θ and B_φ are the components of the transverse amplitude polarized out of the plane of the crack and in the plane of the crack, respectively. The arguments θ and φ specify the position of the transducer with respect to the coordinate frame of the crack (unprimed frame in Fig. 1).

If we rewrite these expressions in terms of the dimensionless parameters $k_i a$ and b/a, we find the relationships

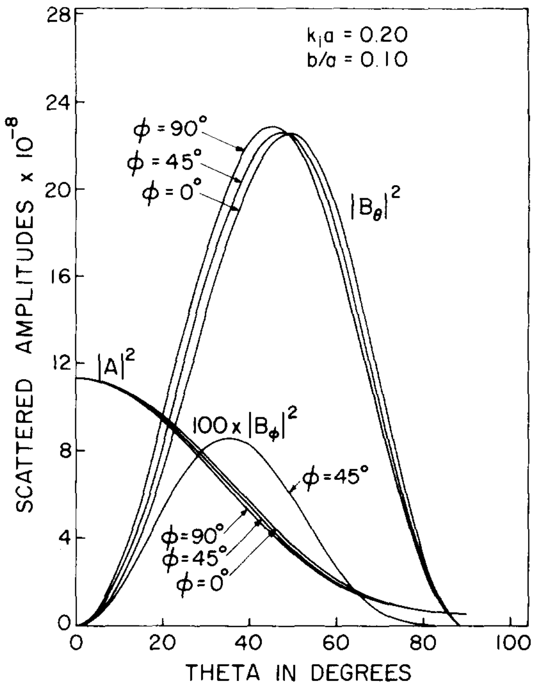

FIG. 2. Scattered squared amplitudes versus θ for the values $k_i a=0.2$, $b/a=0.1$, $\lambda=7.76$, $\mu=4.41$, and $\varphi=0°$, 45°, and 90°. Note that $|B_\varphi|^2=0$ for $\varphi=0°$ and 90°.

$$A(k_i,a,b)=aA(k_i a,1,b/a),$$
$$B_{\theta,\varphi}(k_i,a,b)=aB_{\theta,\varphi}(k_i a,1,b/a). \tag{18}$$

These dimensionless squared amplitudes, $|A(k_i a,1,b/a)|^2$, etc., are plotted versus θ for values of $\varphi=0°$, 45°, and 90°, and a value of $k_i a=0.2$ and $b/a=0.1$, using the elastic constants for titanium $\lambda=7.76$ and $\mu=4.41$ (see Fig. 2). Although we have plotted the amplitudes for the particular values of $k_i a$ and b/a given, one can show that the shapes of these curves will stay roughly the same as we change these parameters. Furthermore, the relative heights of the $|A|^2$ and $|B_\theta|^2$ curves will remain the same while the ratio of $|B_\varphi|^2/|B_\theta|^2$ decreases to zero as b/a goes to unity. This last observation is a consequence of the fact that the amplitude B_φ is a measure of the asymmetry of the system with respect to rotations about the $\hat{z}$ axis. For a circular crack ($b/a=1$), B_φ must be zero as there is no preferred sense of rotation.

If we divide Eq. (17) by Eq. (16), we see that the ratio B_φ/B_θ goes like $(c_1-c_2)/c_3$. The dependence of this quantity on b/a is shown in Fig. 3.

With these observations we conclude that amplitudes with the basic shapes and relative heights as shown in Fig. 2 are the signature of a crack scatterer. In particular the condition $|A|^2$ maximal and $|B_\theta|^2$, $|B_\varphi|^2=0$ locates the direction of normal incidence, $\theta=0°$. The orientation of the crack within the xy plane can be determined by noting that for any $\theta<45°$, $|B_\theta|^2$ is a minimum for $\varphi=0°$. From Fig. 2, one sees that this measurement is most sensitive for $\theta\approx 30°$.

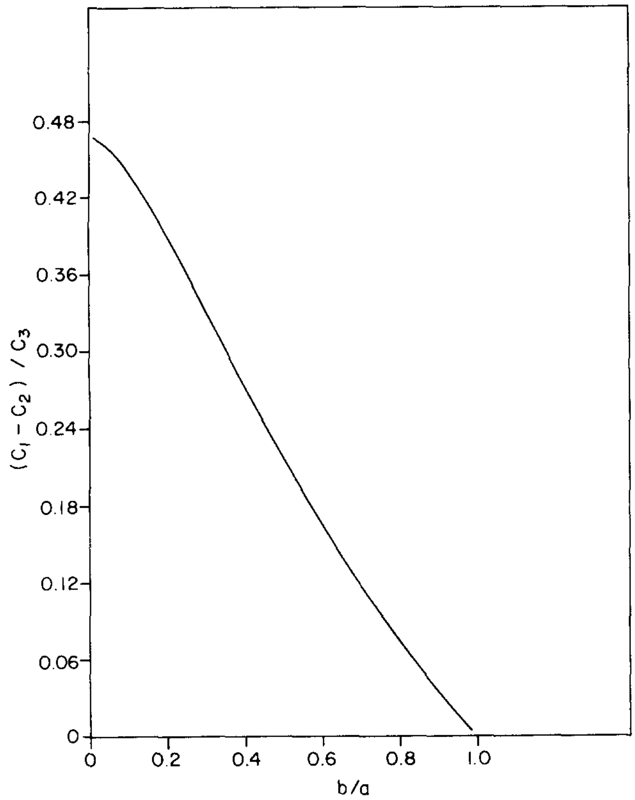

FIG. 3. $(c_1-c_2)/c_3$ plotted versus b/a. $\lambda=7.76$ and $\mu=4.41$.

Having determined the orientation of the crack, we now turn to the determination of the ratio b/a. As mentioned before, the ratio of amplitudes B_φ/B_θ provides a measure of b/a by way of Fig. 3. However, owing to the large difference in heights of these amplitudes, an accurate experimental measurement of this ratio using current techniques might prove difficult. We turn, therefore, to the φ dependence of the amplitudes. The squared amplitudes, $|A(k_i a,1,b/a)|^2$, etc. are plotted versus φ for values of $\theta=30°$, $k_i a=0.2$, and $b/a=0.1$, using $\lambda=7.76$ and $\mu=4.41$ in Fig. 4. The φ dependence of these amplitudes is again a measure of the asymmetry and hence b/a of the crack. For a circular crack ($b/a=1$) with rotational invariance about $\hat{z}$, all amplitudes must be independent of φ. As b/a is changed from unity, an additional oscillation in φ is introduced, the height of which is determined by b/a. Looking in particular at the curve for $|B_\theta|^2$ we can compute from Eq. (16) the ratio $\Delta B_\theta^2/B_{\theta,\min}^2$. For small ka the φ dependence of the $I(\mathbf{k}_i-\boldsymbol{\beta})$ terms may be ignored, so for $\theta=30°$,

$$\frac{\Delta B_\theta^2}{B_{\theta,\min}^2}=\frac{\Delta B}{B_{\min}}\left(2+\frac{\Delta B}{B_{\min}}\right), \tag{19}$$

where

$$\frac{\Delta B}{B_{\min}}=\frac{c_1-c_2}{2c_3[\lambda/\mu+(3/2)]-c_1}.$$

This determines a unique relation between this ratio and b/a (see Fig. 5). As the ratio $\Delta B_\theta^2/B_{\theta,\min}^2$ should be readily measurable (perhaps by means of a differential technique), b/a can thus be determined.

Finally, in order to determine the size of the crack we look at the $k_i a$ dependence of the amplitudes. For small $k_i a$ ($\lesssim 0.4$) the term $I(\gamma)$ in Eqs. (15)–(17) is slowly varying and may be replaced by the value $I(0)$. If this is done, the only $k_i a$ dependence left in the amplitudes is the overall multiplicative factor $(k_i a)^2$ and hence we have,

$$A(k_i,a,b)=aA(k_i a,1,b/a)=a^3A(k_i,1,b/a).$$

Since k_i is known and b/a may be determined as above, $A(k_i,1,b/a)$ may be computed. If an absolute measurement of $A(k_i,a,b)$ can be made, a can then be determined.

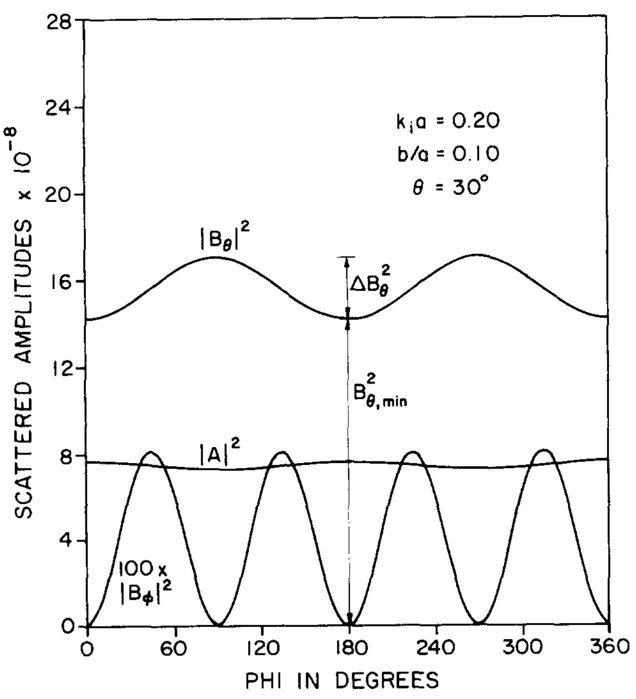

FIG. 4. Scattered squared amplitudes versus φ for the values $\theta=30°$, $k_i a=0.2$, $b/a=0.1$, $\lambda=7.76$ and $\mu=4.41$.

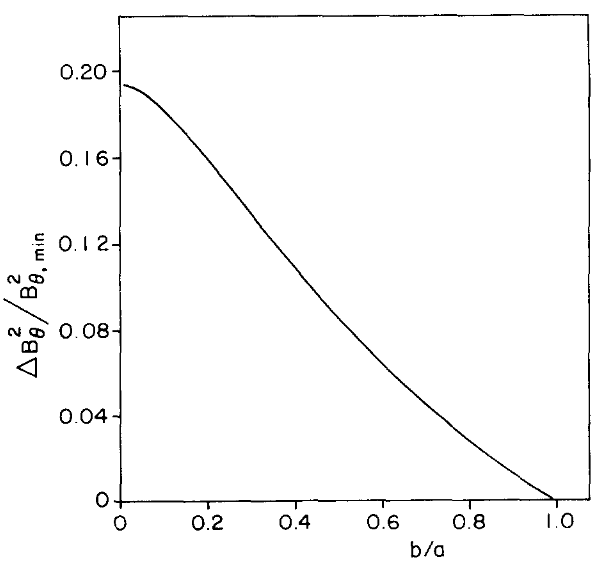

FIG. 5. The ratio $\Delta B_\theta^2/B_{\theta,\min}^2$ versus b/a. $\lambda=7.76$ and $\mu=4.41$.

One can hope to extract the size a from unnormalized measurements by probing at higher $k_i a$ and not making the approximation $I(\gamma)=I(0)$. For example, if one looks at the longitudinal backscattered amplitude at $\theta=90°$ and $\varphi=0°$, and scans through k_i, the quasistatic approximation keeping $I(\gamma)$ predicts a peak at about $k_i a \simeq 1.4$. While is is not clear how good the quasistatic approximation is at this high a $k_i a$, one can expect this condition to at least provide an order-of-magnitude estimate for the size of the crack.

IV. CONCLUSION

In conclusion, we feel that the quasistatic approximation is an accurate and valuable approximation in the long-wavelength limit. Detailed formulas are provided for the far-field scattered amplitudes for several experiment configurations. The particular case of backscattering from an incident longitudinal wave is examined in detail. We summarize the simple procedure the quasistatic approximation yields to determine the characteristics of the elliptical crack from backscattering data for an incident longitudinal wave.

(1) The transducer is moved about until the reflected transverse amplitudes vanish and the reflected longitudinal amplitude is maximal. This locates the direction of normal incident, i.e., $\theta=0°$, and determines the plane of the crack. Knowing the plane of the crack one can now define the directions for the in-plane (B_φ) and out-of-plane (B_θ) transverse polarizations.

Move the transducer up to some $\theta<45°$ and scan through the angle φ. $|B_\theta|^2$ will be minimum when $\varphi=0°$. Thus the orientation of the crack is determined.

(3) Move the transducer up to $\theta=30°$ and scan through the angle φ to determine the ratio $\Delta B_\theta^2/B_{\theta,\min}^2$. Figure 5 (with λ and μ appropriate to the material in question) then yields the ratio b/a and the eccentricity of the crack is known.

(4) If the absolute intensity $|A(k_i,a,b)|^2$ can be measured, compute the amplitude $|A(k_i,1,b/a)|^2$ from Eq. (15). The relation

$$\frac{|A(k_i,a,b)|^2}{|A(k_i,1,b/a)|^2}=a^6$$

then determines a.

Thus even in the long-wavelength region where imaging of the scatterer is impossible, one can deduce in a simple way all information regarding the nature of the elliptical crack.

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