Universality of Jamming Criticality in Overdamped Shear-Driven Frictionless Disks

Daniel Vågberg,[1] Peter Olsson,[1] and S. Teitel[2]
[1]Department of Physics, Umeå University, 901 87 Umeå, Sweden
[2]Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627, USA
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We investigate the criticality of the jamming transition for overdamped shear-driven frictionless disks in two dimensions for two different models of energy dissipation: (i) Durian's bubble model with dissipation proportional to the velocity difference of particles in contact, and (ii) Durian's "mean-field" approximation to (i), with dissipation due to the velocity difference between the particle and the average uniform shear flow velocity. By considering the finite-size behavior of pressure, the pressure analog of viscosity, and the macroscopic friction σ/p, we argue that these two models share the same critical behavior.

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Many different physical systems, such as granular materials, suspensions, foams, and emulsions, may be modeled in terms of particles with short ranged repulsive contact interactions. As the packing fraction ϕ of such particles is increased, the system undergoes a jamming transition from a liquid state to a rigid but disordered solid. It has been proposed that this jamming transition is a manifestation of an underlying critical point, "point J," with associated scaling properties such as is found in equilibrium phase transitions [1,2]. Scaling properties are indeed found when such systems are isotropically compressed, with pressure, elastic moduli, and contact number increasing as power laws as ϕ increases above the jamming ϕ_J [3]. When such systems are sheared with a uniform strain rate $\dot{\gamma}$, a unified critical scaling theory has successfully described both the vanishing of the yield stress as $\phi \to \phi_J$ from above, the divergence of the shear viscosity as $\phi \to \phi_J$ from below, and the nonlinear rheology exactly at $\phi = \phi_J$ [4].

One of the hallmarks of equilibrium critical points is the notion of *universality*; the critical behavior, specifically the exponents describing the divergence or vanishing of observables, depend only on the symmetry and dimensionality of the system, and not on details of the specific interactions. For statically jammed states created by compression, where only the elastic contact interaction comes into play, it is understood that the relevant critical exponents are simply related to the form of the elastic interaction, and are all simple rational fractions [3]. In contrast, shear-driven steady states are formed by a balance of elastic and dissipative forces, and it is thus an important question whether or not the specific form taken for the dissipation is crucial for determining the critical behavior.

In a recent work by Tighe *et al.* [5], it was claimed that changing the form of the dissipation can indeed alter the nature of the criticality for sheared overdamped frictionless disks. In contrast to an earlier work [4], where particle dissipation was taken with respect to a uniformly sheared background reservoir, Tighe *et al.* used a collisional model for dissipation. They argued that this change in dissipation resulted in dramatically different behavior from that found previously, specifically (i) there is no length scale ξ that diverges upon approaching ϕ_J, and so behavior can be described analytically with a mean-field-type model, (ii) critical exponents are simple rational fractions, and (iii) there is no single critical scaling, but rather several different flow regimes, each with a different scaling. In this work we numerically reinvestigate the model of Tighe *et al.* and present results arguing against these conclusions. In particular, we conclude that the two models have rheology that is characterized by the *same* critical exponents, and so are in the same critical universality class.

We simulate bidisperse frictionless disks in two dimensions, with equal numbers of big and small disks with diameter ratio 1.4, at zero temperature. The interaction of disks i and j in contact is $V_{ij} = k_e \delta_{ij}^2/2$, where the overlap is $\delta_{ij} = r_{ij}/d_{ij} - 1$, with d_{ij} the sum of the disks' radii. The elastic force on disk i is $\mathbf{f}_i^{\mathrm{el}} = -\nabla_i \sum_j V_{ij}$, where the sum is over all particles j in contact with i. We use Lees-Edwards boundary conditions [6] to introduce a time-dependent uniform shear strain $\gamma(t) = \dot{\gamma} t$ in the $\hat{x}$ direction.

We consider two different models for energy dissipation. The first, which we call "contact dissipation" (CD), is the model introduced by Durian for bubble dynamics in foams [7], and is the model used by Tighe *et al.* [5]. Here dissipation occurs due to velocity differences of disks in contact,

$$\mathbf{f}_{\mathrm{CD},i}^{\mathrm{dis}} = -k_d \sum_j (\mathbf{v}_i - \mathbf{v}_j), \qquad \mathbf{v}_i = \dot{\mathbf{r}}_i. \tag{1}$$

In the second, which we call "reservoir dissipation" (RD), dissipation is with respect to the average shear flow of a background reservoir,

$$\mathbf{f}_{\mathrm{RD},i}^{\mathrm{dis}} = -k_d (\mathbf{v}_i - \mathbf{v}_{\mathrm{R}}(\mathbf{r}_i)), \qquad \mathbf{v}_{\mathrm{R}}(\mathbf{r}_i) \equiv \dot{\gamma} y_i \hat{x}. \tag{2}$$

RD was also introduced by Durian [7] as a "mean-field" [8] approximation to CD, and is the model used in many earlier works on criticality in shear-driven jamming [4,8–10].

The equation of motion for both models is

$$m_i \dot{\mathbf{v}}_i = \mathbf{f}_i^{\mathrm{el}} + \mathbf{f}_i^{\mathrm{dis}}. \tag{3}$$

Here we are interested in the overdamped limit, $m_i \to 0$ [7]. In RD it is straightforward to set $m_i = 0$, in which case the equation of motion becomes simply $\mathbf{v}_i = \mathbf{v}_{\mathrm{R}}(\mathbf{r}_i) + \mathbf{f}_i^{\mathrm{el}}/k_d$; we call this limit RD_0. In CD, because the dissipation couples velocities one to another, setting $m_i = 0$ effectively requires inverting the matrix of contacts to rewrite the equation of motion in a form suitable for numerical integration. Instead of that numerically difficult procedure, our approach here is to simulate particles with a finite mass, and verify that the mass is small enough for the system to be in the overdamped $m_i \to 0$ limit; we call this limit CD_0. For our simulations we use units in which $k_e = k_d = 1$, length is in units such that the small disk diameter $d_s = 1$, time is in units of $\tau_0 \equiv k_d d_s^2/k_e = 1$, and particles with equal mass density, such that m_i for a particle of diameter d_i is $m_i = 2m\pi d_i^2/4$, with $m = 1$. In our Supplemental Material [11] we confirm that this choice is sufficient to be in the $m_i \to 0$ limit. For RD_0 our simulations use $N = 65\,536$ particles, while for CD_0 we use $N = 262\,144$, unless otherwise noted. For RD_0 our slowest strain rate is $\dot{\gamma} = 10^{-9}$, while for CD_0 we can reach only $\dot{\gamma} = 10^{-7}$.

Before presenting our evidence that the two models RD_0 and CD_0 have the same critical rheology, we first comment on one quantity that is clearly very different in the two models, the transverse velocity correlation, $g_y(x) \equiv \langle v_y(0) v_y(x) \rangle$. In RD_0 $g_y(x)$ shows a clear minimum at a distance $x = \xi$, and this length diverges as one approaches the critical point (ϕ_J, $\dot{\gamma} \to 0$) [4]. In CD_0 however, it was found [5] that $g_y(x)$ decreases monotonically without any obvious strong dependence on either ϕ or $\dot{\gamma}$. This led Tighe *et al.* to conclude that there is no diverging length ξ in model CD_0, that the only macroscopic length scale is the system length L, and thus there are no critical fluctuations. In our own work we have confirmed this dramatic difference in the behavior of $g_y(x)$, but see our Supplemental Material for further comments [11].

However the apparent absence of a diverging ξ in $g_y(x)$ for CD_0 does not necessarily imply that such a diverging length does not exist. In the following we present evidence for such a diverging ξ in CD_0 by considering the finite-size dependence of the pressure p as a function of strain rate $\dot{\gamma}$ at ϕ_J. By a critical scaling analysis of the pressure analogue of viscosity, $\eta_p \equiv p/\dot{\gamma}$, we further show that the rheology in CD_0 is characterized by the same critical exponents as is RD_0. Finally, we consider the macroscopic friction $\mu \equiv \sigma/p$ in the two models, with σ the shear stress, and show that they behave similarly. There is no sign of the roughly square root vanishing of μ at ϕ_J that would be expected from the model of Tighe *et al.*, but rather μ appears to be finite passing through ϕ_J, as found in recent experiments on foams [12].

Finite-size dependence of pressure.—We consider here only the elastic part of p which is computed from the elastic contact forces in the usual way [3]. If jamming behaves like a critical point, we expect p to obey finite-size scaling at large system lengths L [13],

$$p(\phi, \dot{\gamma}, L) = L^{-y/\nu} \mathcal{P}((\phi - \phi_J) L^{1/\nu}, \dot{\gamma} L^z). \tag{4}$$

Exactly at $\phi = \phi_J$ the above becomes [14]

$$p(\phi_J, \dot{\gamma}, L) = L^{-y/\nu} \mathcal{P}(0, \dot{\gamma} L^z). \tag{5}$$

For sufficiently small L, where $\dot{\gamma} L^z \ll 1$, we get $p \sim L^{-y/\nu}$. For sufficiently large L, where $\dot{\gamma} L^z \gg 1$, p becomes independent of L and so $p \sim \dot{\gamma}^{y/z\nu}$. The crossover occurs when $L = \xi$ at $\dot{\gamma} L^z \approx 1 \Rightarrow \xi \sim \dot{\gamma}^{-1/z}$, giving a diverging correlation length as $\dot{\gamma} \to 0$.

In Fig. 1(a) we plot p vs L for RD_0 and CD_0 at $\phi = 0.8433 \approx \phi_J$ for several different $\dot{\gamma}$. Both models clearly behave similarly. To determine the crossover ξ we fit our data to the simple empirical form $p = C(1 + [\xi/L]^x)$ that interpolates between the two asymptotic limits. This fit gives the solid lines in Fig. 1(a). The resulting ξ is plotted in Fig. 1(b). We see that ξ is essentially identical in the two models, growing monotonically as $\dot{\gamma}$ decreases, reaching values as large as $\xi \simeq 20$ for our smallest $\dot{\gamma}$. Such a large length, many times the microscopic length set by the particle size, is clear evidence for cooperative behavior [15]. Thus, our results indicate a growing macroscopic length scale in CD_0, just as was found for RD_0. The exponent $z \approx 5.6$ found in Fig. 1(b) must, however, be viewed with caution since corrections to scaling are large at the sizes L considered here [16], and the neglect of such corrections can skew the resulting effective exponents away

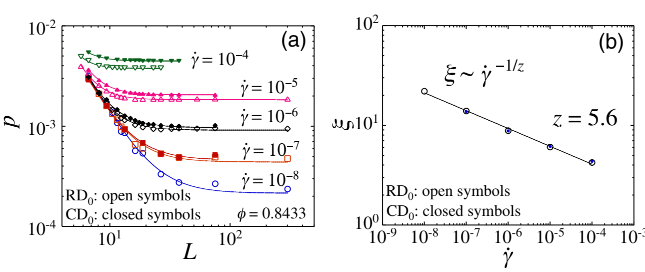

FIG. 1 (color online). (a) Finite-size behavior of pressure p in model RD_0 (open symbols) and model CD_0 (closed symbols) at different strain rates $\dot{\gamma}$. For $\dot{\gamma} = 10^{-8}$ we only have results for model RD_0. The crossover from power law behavior at small L to a constant at large L determines the correlation length ξ, plotted vs $\dot{\gamma}$ in (b). We see that ξ is essentially identical in both models, growing monotonically as $\dot{\gamma}$ decreases, reaching values as large as $\xi \approx 20$ for our smallest $\dot{\gamma}$. As L varies, the number of particles varies from $N = 24$ to 4096.

from their true values at criticality. See our Supplemental Material [11] for a more in depth discussion.

Pressure analog of viscosity.—We now seek to compute the critical exponents of the two models, RD_0 and CD_0, to see if they are indeed in the same universality class. To do this we consider data at various packing fractions ϕ and strain rates $\dot{\gamma}$ close to the jamming transition. We use system sizes large enough ($N = 65\,536$ for RD_0 and $N = 262\,144$ for CD_0), such that finite-size effects are negligible for the data presented here. As in our recent work on RD_0 [13], we consider here the pressure analog of viscosity $\eta_p \equiv p/\dot{\gamma}$, since corrections to scaling are smaller for p than for shear stress σ [13].

To extract the jamming fraction ϕ_J and the critical exponents we use a mapping from our system of soft-core disks to an effective system of hard-core disks. We have previously shown [17] this approach to give excellent agreement with results from a more detailed two variable critical scaling analysis [13] for RD_0. We use it here because it requires no parametrization of an unknown crossover scaling function and so is better suited particularly to CD_0 where the range of our data is more limited ($10^{-7} \le \dot{\gamma}$) as compared to RD_0 ($10^{-9} \le \dot{\gamma}$).

This method assumes that the soft-core disks at ϕ and $\dot{\gamma}$ can be described as effective hard-core disks at $\phi_{\text{eff}}(\dot{\gamma})$, by modeling overlaps as an effective reduction in particle radius [17]. Measuring the overlap via the average energy per particle E, we take

$$\phi_{\text{eff}} = \phi - cE^{1/2y}, \tag{6}$$

where y is the exponent with which the pressure rises as ϕ increases above ϕ_J along the yield stress curve $\dot{\gamma} \to 0$, as in Eq. (4), and c is a constant. We can then express the viscosity of this effective hard-core system as

$$\eta_p(\phi, \dot{\gamma}) = \eta_p^{\text{hd}}(\phi_{\text{eff}}) = A(\phi_J - \phi_{\text{eff}})^{-\beta}. \tag{7}$$

Our analysis then consists of adjusting ϕ_J, the exponents y and β, and the constants c and A in Eqs. (6) and (7), to get the best possible fit to our data.

In Fig. 2 we show the results of such an analysis for CD_0. Panel (a) shows our raw data for η_p vs ϕ, for several different $\dot{\gamma}$, in the narrow density interval around ϕ_J that is used for the analysis. Panel (b) shows the result from fitting η_p for $\dot{\gamma} \le 10^{-6}$ to Eq. (7). Our fitted values $\phi_J = 0.8434$, $\beta = 2.5 \pm 0.2$, and $y = 1.07 \pm 0.05$ for CD_0 are all very close to our earlier results for RD_0 ($\phi_J = 0.8433$, $\beta = 2.58 \pm 0.10$, $y = 1.09 \pm 0.01$) [17] thus suggesting that the critical behavior in CD_0 is the same as in RD_0.

In quoting the fitted values of ϕ_J, β, and y we note that our results for RD_0 include data to much lower strain rates $10^{-9} \le \dot{\gamma}$ as compared to CD_0 where $10^{-7} \le \dot{\gamma}$. For a more accurate comparison of the two models, we should fit our data over the same range of strain rates $\dot{\gamma}$. We therefore carry out a fitting to Eqs. (6) and (7) using data in the interval $\dot{\gamma}_{\min} \le \dot{\gamma} \le \dot{\gamma}_{\max}$. In Fig. 3 we show our results for ϕ_J and β, where we plot the fitted values vs $\dot{\gamma}_{\min}$ for two different fixed values of $\dot{\gamma}_{\max} = 1 \times 10^{-6}$ and 2×10^{-6}. For RD_0 we can extend this procedure down to $\dot{\gamma}_{\min} = 10^{-9}$, while for CD_0 we are limited to $\dot{\gamma}_{\min} = 10^{-7}$. We see that for equivalent ranges of $\dot{\gamma}$, the fitted values of β agree nicely for the two models, for the smaller value of $\dot{\gamma}_{\max}$. We see that ϕ_J for CD_0 is just slightly higher than for RD_0. We cannot say whether this is a systematically significant difference, or whether ϕ_J would decrease slightly to match the value found for RD_0 were we able to study CD_0 down to comparably small $\dot{\gamma}$. Whether or not the ϕ_J of the two models are equal, or just slightly different, the equality of the exponents β strongly argues that models RD_0 and CD_0 are in the same critical universality class.

To return to the results of Tighe *et al.* [5], we note that their value $\phi_J = 0.8423$ for CD_0 is clearly different from our above value of 0.8434. We believe that this difference is due to two main effects: (i) their data are restricted to $10^{-5} \le \dot{\gamma}$ and so do not probe as close to the critical point as we do here, and (ii) their analysis was based on the scaling of shear viscosity $\eta \equiv \sigma/\dot{\gamma}$ rather than the pressure viscosity η_p. As we have noted previously [13] corrections to scaling for σ are significantly larger than they are for p, and without taking these corrections into account, one generally finds a lower value for ϕ_J, such as was also found in the original scaling analysis of RD_0 [4]. Their lower value of ϕ_J, and their higher window of strain rates $\dot{\gamma}$, we believe are also responsible for the different value they find for the exponent describing nonlinear rheology exactly at ϕ_J, $\sigma \sim p \sim \dot{\gamma}^q$; they claim $q = 1/2$ whereas our present result finds a clearly different value $q = y/(\beta + y) \approx 0.30$ [17].

Macroscopic friction.—Finally we consider the macroscopic friction, $\mu \equiv \sigma/p$. In Fig. 4 we plot μ vs ϕ for several different values of strain rate $\dot{\gamma}$. We also show results from quasistatic simulations [16,18], representing the $\dot{\gamma} \to 0$ limit [19]. We use a system with $N = 1024$ particles to more explicitly compare with the results of Tighe *et al.*,

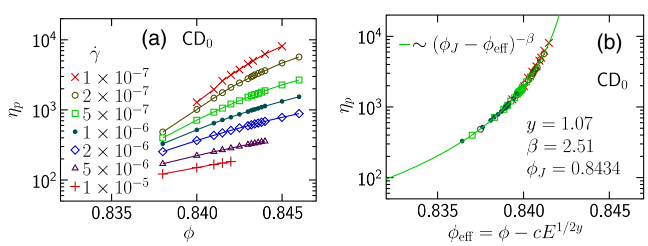

FIG. 2 (color online). Pressure analog of viscosity $\eta_p \equiv p/\dot{\gamma}$ for model CD_0. Panel (a) shows the raw data for $\dot{\gamma} = 10^{-7}$ through 10^{-5} in a narrow interval of ϕ about ϕ_J. Solid lines interpolate between the data points. Panel (b) shows our scaling collapse of η_p vs ϕ_{eff}, with ϕ_J, β, and the parameters in Eqs. (6) and (7) determined from the analysis of data with $\dot{\gamma} \le 10^{-6}$. The solid line is the fitted power law scaling function.

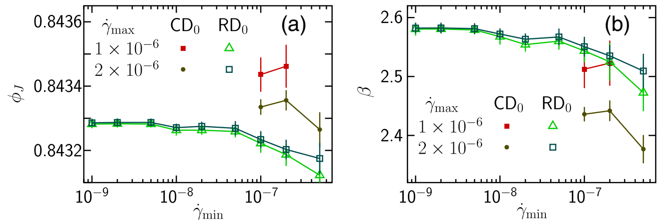

FIG. 3 (color online). Comparison of critical parameters in models RD_0 and CD_0 for similar ranges of strain rate $\dot{\gamma}$. Panel (a) shows the jamming packing fraction ϕ_J and panel (b) the viscosity exponent β that result from fits of our data to Eqs. (6) and (7), for different ranges of $\dot{\gamma}_{\min} \le \dot{\gamma} \le \dot{\gamma}_{\max}$ and $0.838 \le \phi \le 0.846$. The error bars in the figure represent only the statistical errors from the fitting procedure; quoted errors in the text include rough estimates of systematic errors, such as arise when varying the window in ϕ of the data utilized in the fit. We plot results vs varying $\dot{\gamma}_{\min}$ for two different fixed values of $\dot{\gamma}_{\max} = 1 \times 10^{-6}$ and 2×10^{-6}. The open symbols are results for RD_0 and the closed symbols are for CD_0. For RD_0 we have data down to $\dot{\gamma}_{\min} = 10^{-9}$; however, for CD_0 our data go down to only $\dot{\gamma}_{\min} = 10^{-7}$.

who used a similar size system. While μ for the models RD_0 and CD_0 differ slightly at the lower ϕ, we see that near ϕ_J they become essentially equal at the smaller $\dot{\gamma}$, and both RD_0 and CD_0 approach the quasistatic limit as $\dot{\gamma} \to 0$. We thus conclude that μ is finite as ϕ passes through ϕ_J, consistent with recent experiments on foams by Lespiat *et al.* [12].

From our fit to η_p in Fig. 2 we conclude that for both models CD_0 and RD_0 the pressure along the yield stress line, i.e., $\dot{\gamma} \to 0$, $\phi > \phi_J$, vanishes upon approaching ϕ_J as $p_0 \sim (\phi - \phi_J)^y$ with $y \simeq 1.08$. Our results in Fig. 4 then argue that the shear stress along the yield stress line, σ_0, vanishes similarly, so that μ stays finite. However, the prediction of Tighe *et al.* is that the yield shear stress vanishes as $\sigma_0 \sim (\phi - \phi_J)^{3/2}$. Were this conclusion correct, we would expect $\mu \sim (\phi - \phi_J)^{0.42}$, *vanishing* as $\phi \to \phi_J$

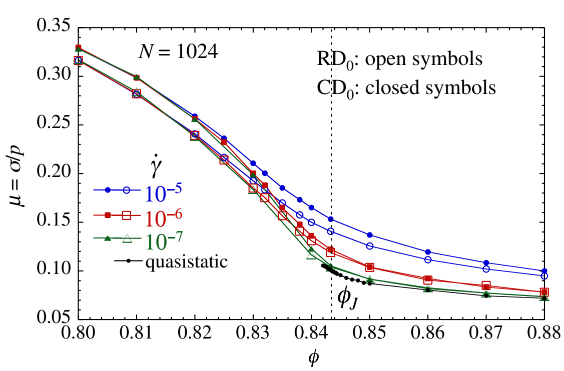

FIG. 4 (color online). Macroscopic friction σ/p vs ϕ for different $\dot{\gamma}$ for models RD_0 (open symbols) and CD_0 (closed symbols), for a system with $N = 1024$ particles. Also shown are results from quasistatic simulations representing the $\dot{\gamma} \to 0$ limit.

from above. Nothing in Fig. 4, where we see that $\mu = \sigma/p$ is a monotonically *increasing* function as ϕ *decreases*, suggests any such vanishing of $\mu(\phi_J)$. We thus conclude from Fig. 4 that the predicted scaling of σ_0 by Tighe *et al.* is not correct, and moreover the two models CD_0 and RD_0 behave qualitatively the same for both pressure p and shear stress σ.

To conclude, we have examined the issue of the universality of the jamming transition for overdamped shear-driven frictionless soft-core disks in two dimensions. We have considered two different dissipative models that have been widely used in the literature: the collisional Durian bubble model CD_0 and its mean-field approximation RD_0. Contrary to previous claims [5], we find clear evidence that CD_0 does exhibit a growing macroscopically large length ξ that appears to diverge as the jamming critical point is approached. We further provide strong evidence that CD_0 and RD_0 are in fact in the same universality class with the same critical exponents at jamming, and have qualitatively the same rheological behavior more generally.

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[19] Quasistatic simulations only accurately compute σ/p for jammed states, where σ and p are both finite. The contribution from flowing, unjammed states, where $\sigma/p = \eta/\eta_p$ (where $\eta = \sigma/\dot{\gamma}$ is the shear viscosity), is not properly accounted for in a naive calculation since σ and p separately vanish as $\dot{\gamma} \to 0$ and so do not contribute to the average. Hence, we only show quasistatic results for $0.842 \leq \phi$, where for $N = 1024$ we find that roughly half or more of the configurations sampled during quasistatic shearing are jammed [16]. We note that the model of Tighe *et al.* [5] applies only above ϕ_J, and so is in the region where the quasistatic calculation is valid.

Universality of Jamming Criticality in Overdamped Shear-Driven Frictionless Disks
Supplemental Material

Daniel Vågberg,[1] Peter Olsson,[1] and S. Teitel[2]
[1]*Department of Physics, Umeå University, 901 87 Umeå, Sweden*
[2]*Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627*
(Dated: June 30, 2014)

TRANSVERSE VELOCITY CORRELATION FUNCTION

The one quantity for which models RD_0 and CD_0 are clearly different is the transverse velocity correlation function, $g_y(x) \equiv \langle v_y(0)v_y(x)\rangle$. Defining the normalized correlation, $G_y(x) \equiv g_y(x)/g_y(0)$, we plot in Fig. 1(a) $G_y(x)$ vs x, for several different values of strain rate $\dot\gamma$, for model RD_0 at $\phi = 0.8433 \approx \phi_J$ in a system of $N = 4096$ particles. We see that $G_y(x)$ has a clear minimum at a distance $x = \ell$, and that ℓ increases as $\dot\gamma \to 0$ and one approaches the critical point. In Ref. [1] ℓ was interpreted as the diverging correlation length ξ. In CD_0 however, it was found [2] that $G_y(x)$ decreases monotonically without any obvious strong dependence on either ϕ or $\dot\gamma$. In Fig. 1(b) we plot $G_y(x)$ vs x, for several different $\dot\gamma$, at $\phi = 0.8433 \approx \phi_J$ in a system of $N = 4096$ particles, confirming this result.

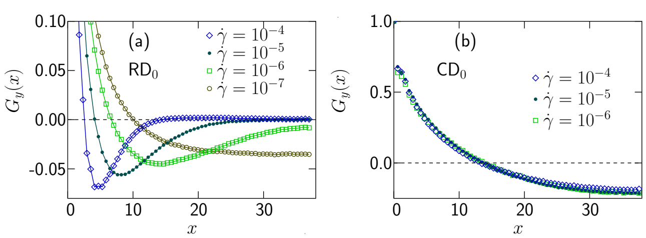

FIG. 1. Normalized transverse velocity correlation function $G_y(x) = g_y(x)/g_y(0)$ at $\phi = 0.8433 \approx \phi_J$ for a system of $N = 4096$ particles. Panel (a) is for model RD_0 with shear rates $\dot\gamma = 10^{-7}$ through 10^{-4}. Panel (b) for model CD_0 at shear rates $\dot\gamma = 10^{-6}$, through 10^{-4}.

As an alternative way to consider the difference in this correlation between the two models, we now consider the Fourier transformed correlation $g_y(k_x) = \int dx\, g_y(x)\mathrm{e}^{ik_x x}$, which we show in Figs. 2(a) and 2(b) for RD_0 and CD_0 respectively at packing fraction $\phi = 0.8433 \approx \phi_J$. For RD_0 we see a maximum in $g_y(k_x)$ at a k_x^* that decreases for decreasing $\dot\gamma$; $\ell \sim 1/k_x^*$ gives the corresponding minimum of the real-space correlation. For CD_0 we show results only for the single strain rate $\dot\gamma = 10^{-6}$ since from Fig. 1(a) we expect no observable difference as $\dot\gamma$ varies. We see an algebraic divergence $g_y(k_x) \sim k_x^{-1.3}$ as $k_x \to 0$. It is this algebraic divergence that causes the real space $G_y(x)$ in CD_0 to become solely a function of x/L as the system length L increases, as was reported in Ref. [2].

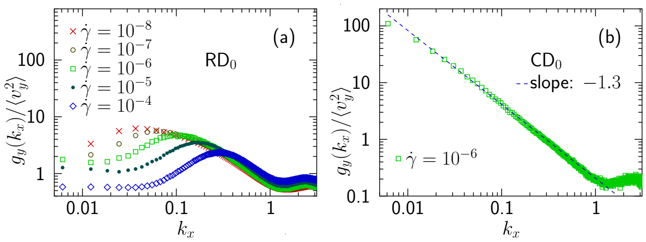

FIG. 2. Fourier transform of the transverse velocity correlation function $g_y(k_x)$ at $\phi = 0.8433 \approx \phi_J$. Panel (a) is for model RD_0 with shear rates $\dot\gamma = 10^{-8}$ through 10^{-5}. The peak in $g_y(k_x)$, moving to smaller k_x as $\dot\gamma$ decreases, is related to the minimum in the real space $g_y(x)$ moving to larger x. The algebraic behavior in panel (b) for model CD_0 at $\dot\gamma = 10^{-6}$, is consistent with the absence of any apparent length scale, as reported in Ref. [2]. The number of particles in these figures are $N = 262144$ except for the two smallest shear rates for RD_0 for which $N = 65536$.

To try and give a qualitative understanding of this differing behavior of $g_y(k_x)$, we can consider how energy is dissipated in each model. In RD_0 the dissipation is $(1/N)\sum_i\langle|\delta\mathbf{v}_i|^2\rangle \approx \int d\mathbf{k}\langle\delta\mathbf{v}(\mathbf{k})\cdot\delta\mathbf{v}(-\mathbf{k})\rangle$. For CD_0, however, the dissipation is $(1/N)\sum_{i,j}\langle|\mathbf{v}_i - \mathbf{v}_j|^2\rangle \approx \int d\mathbf{k}\langle\delta\mathbf{v}(\mathbf{k})\cdot\delta\mathbf{v}(-\mathbf{k})\rangle|\mathbf{k}|^2$, where the sum is over only neighboring particles i, j in contact. Here $\delta\mathbf{v}$ is the non-affine part of the particle velocity. If we make an equipartition-like ansatz, and assume that as $k \to 0$ all modes $\mathbf{k}$, and both spatial directions x, y, contribute equally to the dissipation, we would then conclude that for RD_0 $\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle \propto$ constant, while for CD_0 $\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle \propto 1/k^2$. Noting that $g_y(k_x) = \int dk_y\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle$, we then conclude that for RD_0 we have $g(k_x) \propto$ constant as $k_x \to 0$, while for CD_0 we have the divergence $g(k_x) \propto 1/k_x$. This saturation of $g_y(k_x)$ for RD_0, as compared to the algebraic divergence of $g_y(k_x)$ for CD_0, is what is qualitatively seen in Fig. 2.

The physical reason for this dramatic difference can be viewed as follows. For CD_0, since the dissipation depends only on velocity differences, uniform translation of a large cluster of particles with respect to the ensemble average flow has little cost, thus enabling long wavelength fluctuations. For RD_0 the dissipation is with respect to a fixed background, so uniform translation of a large cluster causes dissipation that scales with the cluster size; long wavelength fluctuations are suppressed.

That the observed divergence in CD_0 is $\sim k_x^{-1.3}$ rather than the simple k_x^{-1} predicted above, suggests that our equipartition ansatz is not quite correct, and that the different modes interact in a non-trivial way. That the exponent of this divergence is not an integer or simple rational fraction suggests the signature of underlying critical fluctuations, even though the correlation $g_y(x)$ itself does not yield any obvious diverging length scale.

FINITE-SIZE-SCALING OF PRESSURE

In Fig. 1 of the main article we showed data for the dependence of pressure p on system size L at different strain rates $\dot\gamma$, at the jamming fraction $\phi_J \approx 0.8433$. We argued that these results provided evidence for a similar growing, macroscopically large, correlation length ξ in both models RD_0 and CR_0. Here we attempt a finite-size-scaling analysis of this data. We must note at the outset, however, that our earlier work [3] demonstrated that it is important to consider corrections-to-scaling to get accurate values for the exponents at criticality, and that corrections-to-scaling are in fact large at the smaller sizes L considered in Fig. 1 of the main article [4]. Since our data for $p(L)$ is not extensive enough to try a scaling analysis including corrections-to-scaling, our results based on a fit to Eq. (5) must be viewed as providing only *effective* exponents describing the data over the range of parameters considered, rather than the true exponents asymptotically close to criticality. We restate Eq. (5),

$$p(\phi_J, \dot\gamma, L) = L^{-y/\nu}\,\mathcal{P}(0, \dot\gamma L^z). \tag{1}$$

We can equivalently write the above in the form

$$p(\phi_J, \dot\gamma, L) = \dot\gamma^{y/z\nu} f(L\dot\gamma^{1/z}), \tag{2}$$

using $f(x) \equiv x^{-y/\nu}\,\mathcal{P}(0, x^z)$. We can now adjust the parameters $q \equiv y/z\nu$ and z to try and collapse the data to a single common scaling curve. Plotting $p/\dot\gamma^q$ vs $L\dot\gamma^{1/z}$ we show the results for RD_0 and CD_0 in Figs. 3(a) and (b). For RD_0 we find the effective exponents $z = 6.5$ and $q = 0.290$, while for CD_0 we find $z = 6.0$ and $q = 0.317$. The values of z found in the present analysis are comparable to the value $z = 5.6$ found in the cruder analysis in Fig. 1(b) of the main article. Note that for both models the scaling function $f(x) \to$ constant as $x \to \infty$, which gives $p \sim \dot\gamma^q$, $q \equiv y/z\nu$, in the limit of an infinite sized system.

The closeness of these fitted *effective* exponents for the two models is one more piece of evidence that RD_0 and CD_0 behave qualitatively the same, and do not have dramatically different rheology as was claimed by Tighe et al. in Ref. [2].

Finally we consider how the effective exponents found here compare to the true exponents asymptotically close to criticality. From our most accurate analysis [3] of the critical behavior in RD_0, using a large system size $N = 65536$ and including the leading corrections-to-scaling, we have found the critical exponents $q = y/z\nu = 0.28 \pm 0.02$ and $y = 1.08 \pm 0.03$, yielding $z\nu = 3.9 \pm 0.4$. This value of q is in reasonable agreement with that found above from the finite-size-scaling analysis of $p(\phi_J, \dot\gamma, L)$. If we take the value of $z \approx 6$ found in the finite-size-scaling analysis, we would then conclude $\nu \approx 0.65$. We note that earlier scaling analyses [1, 5] that similarly ignored corrections-to-scaling found similar values for ν. However our recent [4] more detailed finite-size-scaling analysis of the correlation length exponent, which included corrections-to-scaling, found that $\nu \approx 1$, therefore implying $z \approx 3.9$ as the true critical value. We thus conclude that the larger than expected value of z found here from the finite-size-scaling of p is due to the strong corrections-to-scaling that effect the correlation length at small L.

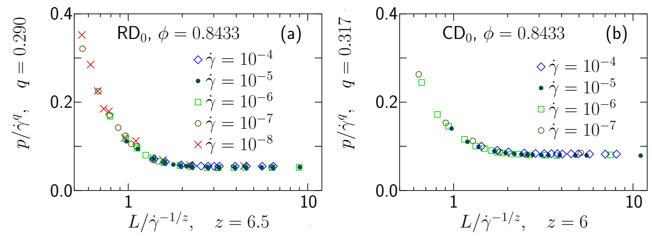

FIG. 3. Scaling collapse of pressure according to Eq. (2) for models RD_0 and CD_0.

As another way to see the effect of corrections-to-scaling on the correlation length, in Fig. 4 we plot our results for p vs L at $\phi = 0.8433 \approx \phi_J$, as obtained from quasistatic simulations [4, 6] representing the $\dot\gamma \to 0$ limit. From Eq. (1) we expect as $\dot\gamma \to 0$ the behavior, $p \sim L^{-y/\nu}$. If we fit the data at small L in Fig. 4 to a power law, we then find the exponent, $y/\nu \approx 1.79$. Using $y = 1.08$ this then gives $\nu \approx 0.60$, in rough agreement with the value of ν obtained from the measured z of our finite-size-scaling of p with $\dot\gamma$. If, however, we fit the data at only the largest L to a power law, we then find the exponent $y/\nu \approx 1.11$. Again using $y = 1.08$, we then get $\nu \approx 0.97$, in better agreement with the expected $\nu \approx 1$. Fig. 4 thus shows in a very direct way that corrections-to-scaling are significant for small system lengths L.

To conclude this section, although our finite-size-scaling of the pressure data in Fig. 1(a) of the main article is effected by corrections-to-scaling, and so gives a larger value for the dynamic exponent z than we believe is actually the case at criticality, nevertheless the correlation length ξ extracted from this data and shown in Fig. 1(b) demonstrates that RD_0 and CD_0 are behaving qualitatively the same, and that both have a macroscopic length scale ξ that is growing (and we would argue diverging) as the jamming transition is approached.

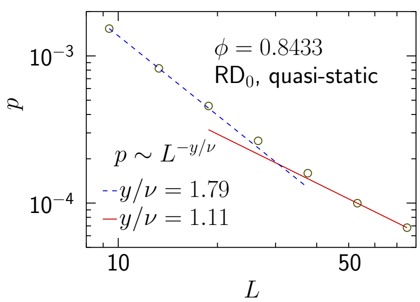

FIG. 4. Pressure p vs system length L at $\phi_J \approx 0.8433$ for quasistatic shearing. Dashed line is a power law fit to the data at the smallest L, giving an exponent $y/\nu \approx 1.79$; solid line is a power law fit to the data at the largest L, giving an exponent $y/\nu \approx 1.11$.

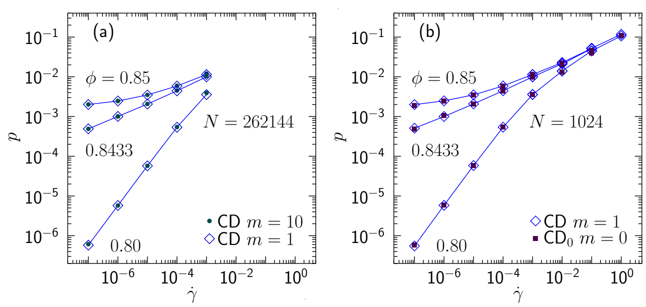

FIG. 5. Pressure p vs. shear strain rate $\dot\gamma$ at packing fractions $\phi = 0.80$, 0.8433, 0.85 for: (a) model CD with $m = 1$ and $m = 10$ for $N = 262144$ particles, and (b) model CD with $m = 1$ and model CD_0 with $m = 0$ for $N = 1024$ particles.

EFFECT OF FINITE MASS ON MODEL CD

We wish to verify that the mass parameter $m = 1$, which we use in model CD, is indeed sufficiently small so as to put our results in the overdamped $m \to 0$ limit corresponding to model CD_0, for the range of parameters studied here. In Fig. 5(a) we show results for the elastic part of the pressure p vs $\dot\gamma$ for model CD, with $N = 262144$ particles, at three different packing fractions: $\phi = 0.80$, $\phi = 0.8433 \approx \phi_J$, and $\phi = 0.85$. We compare results for two different mass parameters, $m = 1$ and $m = 10$. We see that the results agree perfectly for small $\dot\gamma$; significant differences are only found for $\dot\gamma \geq 10^{-3}$ which is higher than the largest shear rate used in our scaling analysis. In Fig. 5(b) we similarly compare results for model CD with $m = 1$ with explicit results for model CD_0, as obtained from simulations using the more costly matrix inversion dynamics for $m = 0$. In this case we are restricted to $N = 1024$ particles because our algorithm for CD_0 scales as N^2. We see that in all cases there is no observed difference between the two models. Thus we conclude that our results from CD with $m = 1$ are indeed in the overdamped $m \to 0$ limit.

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