Josephson-Junction Arrays in Transverse Magnetic Fields

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A class of uniformly frustrated xy models which describe the behavior of a Josephson-junction array in a transverse magnetic field is considered. The frustration f is the fraction of flux quantum of applied field per unit cell of the lattice. The ground-state energy E_0 and critical current are computed for several rational f. The behavior for arbitrary f is discussed, and it is concluded that the resistive transition temperature for $f=p/q$ is bounded by $k_BT_c(p/q) \lesssim \pi|E_0(f)|/2q$. The resistance as a function of temperature and field is determined by the sequence of T_c's thus produced.

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In this Letter we provide a theoretical description of the experiments[1-3] on regular arrays of Josephson junctions which exhibit a rich structure as a function of the applied transverse magnetic field H. The Kosterlitz-Thouless (KT) theory of phase transitions in two-dimensional (2D) systems has been applied to explain uniform superconducting films.[4-6] In a film the vortices form a triangular lattice in the ground state whose melting causes the transition. However, for arrays the lattice introduces a periodic pinning potential which cannot be viewed as a weak perturbation on a uniform film. In the ground state the vortices are constrained to lie at the pinning sites and thus uniform-film treatments are inappropriate. We have studied a class of "uniformly frustrated" xy models which map onto the Josephson-junction array problem. The frustration $f=Ha^2/\Phi_0$ is the fraction of flux quantum Φ_0 of the external field per unit cell of area a^2 of the lattice. In a previous paper[7] we focused primarily on the phase transition in the particular case of $f=\frac{1}{2}$. We showed, using Monte Carlo simulations, that a resistive transition occurs at a finite temperature and that this transition was more complex than the KT transition of the unfrustrated ($f=H=0$) xy model or the KT melting transition of the flux lattice in thin-film superconductors.

In this paper we discuss the case of general f, paying particular attention to the connection to experiments. We exhibit results for the ground-state energy E_0 and the zero-temperature critical currents for various rational values of f. We then present the main result that for rational $f=p/q$ the transition temperature obeys the inequality $k_BT_c(f) \lesssim \pi|E_0(f)|/2q$. We discuss the implications of this result for resistivity measurements, and find good agreement with available data. The detailed nature of the phase transition at various values of f is deferred to a future publication.

Our model is given by

$$\mathcal{H} = -J_0 \sum_{\langle ij\rangle} \cos(\theta_i - \theta_j - \psi_{ij}), \tag{1}$$

where θ_i is the phase of the superconducting node at a site i of a square lattice, $\langle ij\rangle$ denotes near neighbors, and $\psi_{ij} = (2e/\hbar c)\int_i^j \vec{A}_{ij}\cdot d\vec{l}$ is the integral of the vector potential across junction $\langle ij\rangle$. The ψ_{ij} satisfy the constraint that the sum around any unit cell is given by

$$\psi_{ij} + \psi_{jk} + \psi_{kl} + \psi_{li} = 2\pi f = 2\pi Ha^2/\Phi_0, \tag{2}$$

where $\nabla\times\vec{A} = H\hat{z}$ is the external field and a the lattice constant.

We first discuss the applicability of the model [Eq. (1)] to real arrays. The Hamiltonian $\mathcal{H}$ is clearly periodic in f with period 1, and is reflection symmetric about $f=\frac{1}{2}$ on the interval [0, 1]. In experiments, however, this periodicity in f is seen to be modulated by an envelope function. This can be accounted for within the model by noting that the bare coupling J_0 in Eq. (1) will in general be H dependent. This dependence is easy to understand for proximity-coupling junctions. For junctions with an effective area $A \ll a^2$, one expects $J_0(H)$ to have a periodicity HA/Φ_0, corresponding to every time an additional flux quantum threads the gap. Additional effective H dependence may arise as a result of the finite width of the junction and the nonuniform current distribution across it, which is not accounted for in our simple point-junction model. Such effects do not appear to be completely understood.

In addition to depending on H, the bare J_0 will also depend on temperature. In the following we take J_0 constant and thus all our results must be scaled by the known function $J_0(T,H)$ before comparing directly to experiment.[8]

In Eq. (1) we have also assumed that the local field H is equal to the uniform applied field. This will be a good approximation provided that the sample size L is small compared to the transverse penetration depth $\lambda_\perp$.[9] We note in passing that a 2D grid of superconducting wires belongs to the same universality class as the model described above and should display the same qualitative features to be discussed below.

The Hamiltonian (1) maps[7] onto a Coulomb gas problem on a dual lattice with charges $q_n = n - f$, where n is the integer vorticity of the phase variable θ and f is the uniform background charge due to H. The neutral ground state thus consists of a lattice of charges $1-f$ and $-f$. For $f=p/q$ (henceforth p and q are taken with no common factors) we have assumed the ground state to be periodic with unit cell $q\times q$. In Fig. 1 are displayed ground-state lattices for several rational f.

We have studied the $T=0$ properties of the Hamiltonian by doing Monte Carlo simulations on $q\times q$ lattices with periodic boundary conditions. (For several f the assumption of $q\times q$ periodicity was checked by calculating on $nq\times nq$ lattices.) Our results for the ground-state energy E_0 for $f=p/q$, $q\leqslant 8$ are shown in Fig. 2. Once the simulation has determined the location of the charges as in Fig. 1, E_0 may be computed exactly by symmetry arguments applied to the phase differences across bonds. $E_0(f)$ is clearly nonmonotonic on $f\in[0,\frac{1}{2}]$ with sharp features at several f.[10]

We next compute the $T=0$ critical current for an $N\times N$ lattice by considering

$$i(\delta;f)=\frac{2e}{N^2\hbar}J_0\sum_i \sin(\varphi_i-\varphi_{i+\hat{x}}-2\pi f y_i+\delta), \quad (3)$$

where the gauge $\vec{A}=yH\hat{x}$ has been chosen and the φ_i are the phases corresponding to the local minimum of the Hamiltonian

$$\mathcal{H}_\delta=-J_0\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j-A_{ij}+\delta\hat{e}_{ij}\cdot\hat{x})$$

which continuously evolves from the ground state as δ is increased from zero. Again, periodic boundary conditions on θ_i are imposed. This prescription is just a change of variables from the Hamiltonian (1) with the twisted boundary conditions $\theta(x=N)-\theta(x=0)=N\delta$, and puts the system into a metastable state carrying a net current $i(\delta;f)$. The $T=0$ critical current is just the maximum current that the system can carry[11]:

$$i_c(f)=\max_\delta i(\delta;f). \quad (4)$$

In Fig. 3 are plotted $i_c(f)$ obtained from Monte Carlo simulations using $\mathcal{H}_\delta$ for $f=p/q$, $q\leqslant 8$. The large variations in $i_c(f)$ as a function of f should be noted. This feature may be generalized for arbitrary f by considering $i(\delta;f)$ in Eq. (3). Let us observe that $i(\delta;p/q)$ is periodic in δ with period $2\pi/q$ and antisymmetric about π/q in the interval $[0,2\pi/q]$.[12] Next, we note that for $0\leqslant i(\delta)\leqslant i_c$ we have the bound[13] $di/d\delta\leqslant e|E_0(f)|/\hbar$. Combining these two observations with the mean

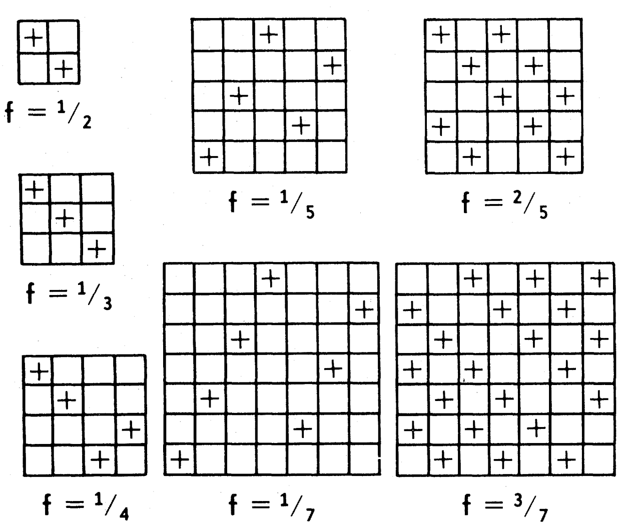

FIG. 1. Ground-state lattices for several rational f. A plus denotes a charge $1-f$ (or equivalently the location of a unit vortex in phase θ), while an empty box denotes a charge $-f$.

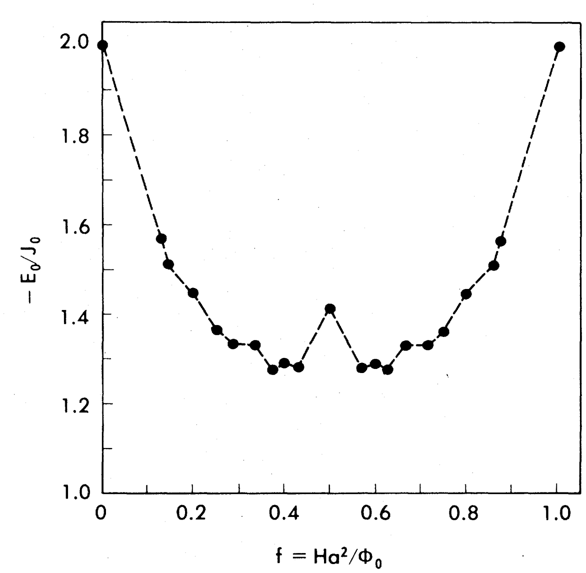

FIG. 2. Ground-state energies for several rational f. (Note that $-E_0$ is plotted.) The dashed line is a guide to the eye only.

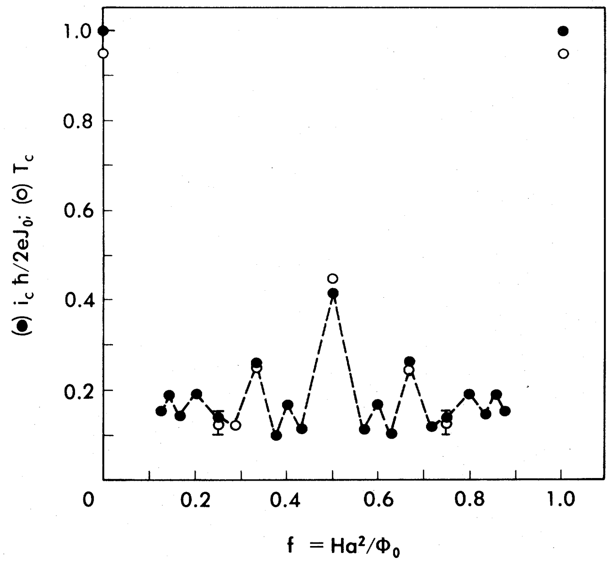

FIG. 3. Zero-temperature critical currents i_c (denoted by solid circles) and zero-current critical temperature T_c (denoted by open circles) for several rational f. The error bars at $f=\frac{1}{4}$ are the characteristic error in our computed T_c's. The dashed line is a guide to the eye only and is not meant to imply that i_c is a continuous function of f. [In fact, Eq. (5) shows quite the contrary!]

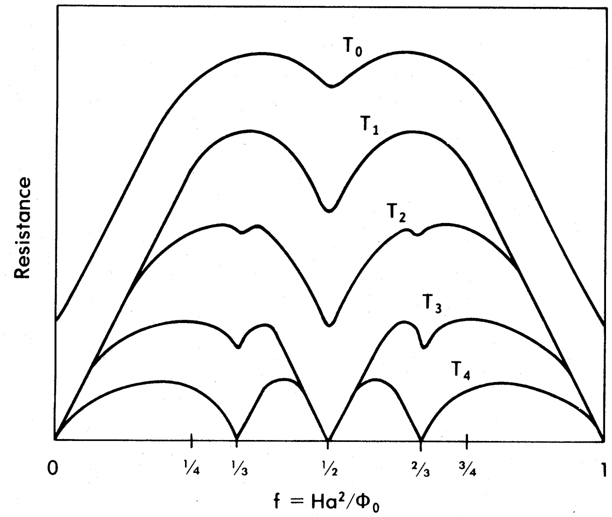

FIG. 4. Schematic plot of resistance vs field $f=Ha^2/\Phi_0$, for a sequence of temperatures: $T_4 < T_c(\frac{1}{3}) < T_3 < T_c(\frac{1}{2}) < T_2 < T_1 < T_c(0) < T_0$. Note that the curves will be periodically extended on the integers f.

value theorem yields the bound on $i_c(f)$:

$$i_c(p/q) \leq [e|E_0(f)|/\hbar]\pi/q. \qquad (5)$$

Thus $i_c(f)$ exhibits dramatic discontinuous variations with f.

Note that $i_c(f)$ of Eq. (5) is the current above which a nonzero voltage first appears. This need not, however, be the current i^* at which the I-V characteristic displays its most rapid variation, rising to saturate at its high-current limit. This latter current i^*, which may be viewed as the experimentally defined critical current, is expected to be of order $i^* \approx (e/\hbar)|E_0(f)|$.

Turning now to the case of zero applied current but finite temperature one expects on energetic grounds that

$$k_B T_c(f) \lesssim (\hbar/2e) i_c(f). \qquad (6)$$

Once the temperature is large enough to produce average current fluctuations in the array of order i_c, vortices will be able to move, and a resistive transition will occur. (The inequality allows for other mechanisms producing a transition earlier.) In Fig. 3 we have plotted on the same graph as i_c of $T=0$, the values of T_c for $f=0$, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ obtained from Monte Carlo simulations as described in Ref. 7. For these cases the assertion (6) is satisfied. Combining Eqs. (5) and (6) produces a bound on $T_c(f)$:

$$k_B T_c(p/q) \lesssim \pi|E_0(f)|/2q. \qquad (7)$$

By the symbol "$\lesssim$" we allow for equality within factors of $O(1)$. It is the dependence of the bound on $1/q$ that is crucial. Suggested by (7) is the conclusion that $T_c(f)=0$ for irrational f.

A simple understanding for the bounds (6) and (7) may be given for the case $f=1/n$, n large. Here the ground state consists of a lattice of charges $q_+ = 1-1/n$ separated by a distance $\sim n^{1/2}a$ from each other. The sum of gauge-invariant phase differences around the unit cell containing q_+ must be $2\pi(1-1/n)$ and so the phase across each bond in this cell is $\frac{1}{2}\pi(1-1/n)$. The ground-state current in each bond is then $(2e/\hbar)J_0 \sin[\frac{1}{2}\pi \times (1-1/n)]$ or almost equal to the critical current of the bare junction. Thus any additional current imposed on the junction will force it to go normal permitting the charge to move. Of course the resistance R due to these moving charges is only of the order of their density, i.e., $1/n$. The resistance will remain low until a temperature $k_B T \approx E_0(f)/2$ is reached at which point additional charge pairs $1-1/n$, $-1-1/n$ proliferate (analogous to the $+1$, -1 charges of the $f=0$, KT picture) driving R to its high-T limit. Similarly, we find in our numerical studies that for $f=p/q$, as q increases the ground-state current configuration induced by the field contains a certain fraction of bonds which carry current increasingly approaching the bare-junction critical current. These bonds are readily driven normal by applied or thermal fluctuating currents and hence cause a reduction in i_c and T_c from values of $f=p/q$ with smaller q.

We now consider the implications of (7) for the array resistance $R(T;f)$. Consider an $f_0=p_0/q_0$ such that $T<T_c(f_0)$, and so $R(T;f_0)=0$. For all $f=p/q$ sufficiently close to f_0, $q \gg q_0$ and hence by (7), $T_c(f)<T$ and $R(T;f)>0$. Since $f \approx f_0$, we may regard the ground state of f as being built of the lattice for f_0 with a superlattice of defects or domain walls to ensure charge neutrality. The charge contained in these defects is just $f-f_0$. Below $T_c(f)$, these defects are pinned. However, when $T_c(f)<T<T_c(f_0)$, the defects unpin and their motion will give rise to nonzero R. As resistance will be proportional to the number of free charges we find for $f \approx f_0$,

$$R(T;f) \propto |f-f_0|. \qquad (8)$$

Of course, inhomogeneities in the experimental system, or fluctuations in the applied H, could round off such a cusp.

Our conclusions are summarized in Fig. 4. To wit, the experimentally measured $R(H)$ for different values of T can be understood in terms of the sequence of critical temperatures $T_c(H)$ for different values of the field. Consider a temperature T_3 in Fig. 4 such that only two f's ($f=0,\frac{1}{2}$) exist such that $T_3<T_c(f)$. The resistance is zero at these values of the field and shows an increase for values around it. When the temperature is increased to T_2, $T_c(0)>T_2>T_c(\frac{1}{2})$, one finds a residual valley in $R(H)$ around $f=\frac{1}{2}$ and it persists for a range of T above $T_c(\frac{1}{2})$. When the temperature is lowered new valleys appear at those H corresponding to the next highest $T_c(f)$. The specific sequence may depend on the lattice structure, etc., but we emphasize the general qualitative behavior of R: At any fixed T, only a finite number of f values will have zero resistance and this number increases as T is lowered. These qualitative features, in particular the valley at $f=\frac{1}{2}$, have been observed in experiments reported in Refs. 1 and 3.

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[12]Consider the transformation $\delta \rightarrow \delta' = \delta + 2\pi/q$ and note that $i(\delta')$ may be recast in the form $i(\delta)$ by shifting the lattice in the y direction, $y_i' = y_i + n$, where n is determined by $np+mq=1$ for integral m. That such integers n and m always exist for coprime p and q is an elementary result in algebra. The antisymmetry about π/q is shown similarly.

[13]Consider the finite-temperature analog of Eq. (3). If $\langle \cdots \rangle$ denotes a thermal average with respect to H_δ and $\langle u_x \rangle$ the average energy in the x bonds, then one finds

$$\frac{d}{d\delta}\langle i(\delta;f)\rangle = -\frac{2e}{\hbar}\langle u_x(\delta;f)\rangle - \frac{\hbar N}{2ek_BT}[\langle i^2(\delta;f)\rangle - \langle i(\delta;f)\rangle^2] \le -\frac{2e}{\hbar}\langle u_x(\delta;f)\rangle \le \frac{e}{\hbar}|E_0(f)|.$$