Positional Disorder in the Fully Frustrated Josephson Junction Array: Random Gaussian Phase Shifts in the Fully Frustrated 2D *XY* Model

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We consider the effect of positional disorder on a Josephson junction array with an applied magnetic field of $f = 1/2$ flux quantum per unit cell. This is equivalent to the problem of random Gaussian phase shifts in the fully frustrated 2D *XY* model. Using simple analytical arguments and numerical simulations, we present evidence that the ground state vortex lattice of the pure model becomes disordered, in the thermodynamic limit, by *any* finite amount of positional disorder. [S0031-9007(99)09518-6]

The stability of vortex lattices to random disorder is a topic of considerable recent interest, motivated by studies of the high temperature superconductors. In two dimensions (2D), periodic arrays of Josephson junctions form a well controlled system for investigating similar issues of vortex fluctuations and disorder. Here we consider the effect of "positional" disorder on the vortex lattice of the fully frustrated Josephson array, with $f = 1/2$ flux quantum of applied magnetic field per unit cell.

Positional disorder [1–4] was first discussed with respect to the Kosterlitz-Thouless (KT) transition for the $f = 0$ model in zero magnetic field. Early arguments [1] predicting a reentrant normal phase at low temperatures have been revised by recent works [2,4] which argue that there is a finite critical disorder strength $\sigma_c \simeq \sqrt{\pi/8}$; for $\sigma < \sigma_c$ an ordered state persists for $0 \leq T \leq T_c(\sigma)$. For the pure $f = 1/2$ case on a square grid [5,6], the ordered state has two broken symmetries: the $U(1)$ symmetry ("KT-like" order) associated with superconducting phase coherence and the $Z(2)$ symmetry ("Ising-like" order) associated with the "checkerboard" vortex lattice, in which a vortex sits on every other site. Previous works [7,8] have considered the effect of positional disorder on this $f = 1/2$ model; all have concluded that both Ising-like and KT-like order persist for at least small disorder strengths σ. In this Letter, however, we present new arguments that suggest that, for $f = 1/2$, the critical disorder is $\sigma_c = 0$.

The Hamiltonian for the Josephson array is given by the "frustrated" 2D *XY* model [6]

$$\mathcal{H}[\theta_i] = \sum_{i\mu} U(\theta_i - \theta_{i+\hat{\mu}} - A_{i\mu}), \qquad (1)$$

where i are the sites of a periodic square grid with basis vectors $\hat{\mu} = \hat{x}, \hat{y}$, the sum is over all nearest neighbor (nn) bonds $\langle i, i + \hat{\mu} \rangle$, and $\theta_i - \theta_{i+\hat{\mu}} - A_{i\mu}$ is the gauge invariant phase difference across the bond, with $A_{i\mu} = (2\pi/\phi_0) \int_i^{i+\hat{\mu}} \mathbf{A} \cdot d\ell$ the integral of the vector potential.

Positional disorder arises from random geometric distortions of the bonds of the grid, resulting in $A_{i\mu} = A_{i\mu}^{(0)} +$

$\delta A_{i\mu}$; $A_{i\mu}^{(0)}$ is the value in the absence of disorder, and $\delta A_{i\mu}$ is the random deviation. We take the $\delta A_{i\mu}$ to be independent Gaussian random variables with

$$[\delta A_{i\mu}] = 0, \quad \text{and} \quad [\delta A_{i\mu} \delta A_{j\nu}] = \sigma^2 \delta_{ij} \delta \mu\nu. \quad (2)$$

$[\cdots]$ denotes an average over the quenched disorder. The positionally disordered array is thus also referred to as the *XY* model with random Gaussian phase shifts.

When $U(\phi)$ is the Villain function [9], the Hamiltonian (1) is equivalent to a dual "Coulomb gas" of interacting vortices [5,10,11],

$$\mathcal{H}[n_i] = \frac{1}{2} \sum_{ij} (n_i - f - \delta f_i) G_{ij} (n_j - f - \delta f_j). \tag{3}$$

The sum is over all pairs of *dual* sites i, j, n_i is the integer vorticity on site i, and the interaction G_{ij} is the Green's function for the 2D discrete Laplacian operator, $\Delta_{ik} G_{kj} = -2\pi \delta_{ij}$, where $\Delta_{ij} \equiv \delta_{i,j+\hat{x}} + \delta_{i,j-\hat{x}} + \delta_{i,j+\hat{y}} + \delta_{i,j-\hat{y}} - 4\delta_{ij}$. For large separations, $G_{ij} \simeq -\ln|\mathbf{r}_i - \mathbf{r}_j|$. The $f_i \equiv f + \delta f_i$ are $(1/2\pi)$ times the circulation of the $A_{i\mu}$ around dual site i; f is the average applied flux, while δf_i is the deviation due to the random $\delta A_{i\mu}$,

$$\delta f_i = \frac{1}{2\pi} [\delta A_{i,x} + \delta A_{i+\hat{x},y} - \delta A_{i+\hat{y},x} - \delta A_{i,y}]. \tag{4}$$

Geometrically distorting a bond increases the flux through the cell on one side of the bond, while reducing the flux through the cell on the opposite side by the same amount. The δf_i are thus *anticorrelated* among nn sites. Positional disorder is thus the same as random dipole pairs of quenched charges $\pm\delta f_i$ [1]. From Eqs. (2) and (4) we get

$$[\delta f_i] = 0, \quad \text{and} \quad [\delta f_i \delta f_j] = -\frac{\sigma^2}{4\pi^2} \Delta_{ij}. \tag{5}$$

The Hamiltonian (3) can be rewritten as interacting charges in a one-body random potential [2],

$$\mathcal{H}[q_i] = \frac{1}{2} \sum_{ij} q_i G_{ij} q_j - \sum_i q_i V_i , \qquad (6)$$

where $q_i \equiv n_i - f$, and the random potential is $V_i = \sum_j G_{ij} \delta f_j$. For $f = 1/2$, $q_i = \pm 1/2$. From Eq. (5),

$$[V_i] = 0 ,$$

and

$$[V_i V_j] = \sum_{k,l} G_{ik} [\delta f_k \delta f_l] G_{lj} \qquad (7)$$

$$= -\frac{\sigma^2}{4\pi^2} \sum_{k,l} G_{ik} \Delta_{kl} G_{lj} = \frac{\sigma^2}{2\pi} G_{ij} ,$$

The V_i thus have logarithmic long range correlations.

We now use an Imry-Ma-type [12] argument to estimate the stability of the doubly degenerate checkerboard ground state to the formation of a square domain of side L. The energy of such an excitation consists of a domain wall term, E_d, which is present for the pure case, and a pinning term, E_p, due to the interaction with the random V_i. $E_d(L)$ has the form [13]

$$E_d \simeq aL + c \ln L + d . \qquad (8)$$

The first term is the interfacial tension of the domain wall; the second term comes from net charge that builds up at the corners of the domain [14]. Calculating $E_d(L)$ numerically for a pure system, we find an excellent fit to Eq. (8), with $a = 0.28$, $c = 0.15$, and $d = 0.058$.

By Eq. (7), the average pinning energy of the domain $\mathcal{D}$, $[E_p] = 2 \sum_{i \in \mathcal{D}} q_i [V_i] = 0$, but the variance is

$$[E_p^2] = 4 \sum_{i,j \in \mathcal{D}} q_i [V_i V_j] q_j$$

$$= \frac{4\sigma^2}{2\pi} \sum_{i,j \in \mathcal{D}} q_i G_{ij} q_j = \frac{4\sigma^2}{\pi} E_0 , \qquad (9)$$

where $E_0 = (\pi/32) L^2$ is the ground state energy of the checkerboard domain [15]. The root mean square (rms) pinning energy is thus

$$[E_p]_{\text{rms}} = bL , \qquad b = \frac{\sigma}{2\sqrt{2}} \simeq 0.35\sigma . \qquad (10)$$

For domains whose energy is *lowered* by the interaction with V_i, the typical excitation energy is $E = E_d - [E_p]_{\text{rms}}$. Equations (8) and (10) imply that when $b > a$, i.e., when $\sigma > \sigma_c \simeq 0.8$, $E(L)$ has a maximum at $L = \xi \equiv (c/2\sqrt{2})/(\sigma - \sigma_c)$. Domains of size $L > \xi$ will lower their energy by increasing in size, and so disorder the system. Thus, one naively expects that when $\sigma < \sigma_c$ the system preserves its Ising-like order, but when $\sigma > \sigma_c$ the system is disordered into domains of typical size ξ.

However, the leading size dependencies of Eqs. (8) and (10), $E_d \sim [E_p]_{\text{rms}} \sim L$, are exactly the same as found in the 2D nn random field Ising model (RFIM). For the RFIM it is known [12,16,17] that 2D is the lower critical dimen-

sion, that the randomness causes domains walls at $T = 0$ always to roughen and so acquire an effective *negative* line tension, and that the critical disorder is $\sigma_c = 0$; i.e., *any amount of disorder, no matter how weak, destroys the Ising-like order of the pure case*. By analogy, we suggest that the positionally disordered $f = 1/2$ 2D XY model similarly has $\sigma_c = 0$. A very similar approach was previously used by Benedict and Moore [18] to argue that $\sigma_c = 0$ in the $f = 1/2$ model with *uncorrelated* random δf_i.

To check our prediction, we carry out Monte Carlo (MC) simulations of the Hamiltonian (3) with periodic boundary conditions on $L \times L$ square grids. Our MC procedure is as follows [15]. One MC excitation attempt consists of the insertion of a neutral $n = \pm 1$ vortex pair on nn or next nn sites, which is accepted or rejected using the usual Metropolis algorithm. L^2 such attempts we call one MC pass. At each temperature we typically used 4000 MC passes to equilibrate the system, followed by 128 000 MC passes to compute averages. Every 100 passes we attempt a global excitation reversing the sign of all the charges, $q_i \rightarrow -q_i$. For each disorder realization we cooled down two distinct "replicas," starting with different random charge configurations and using different random number sequences. In only about 3% of the cases did the two replicas fail to give reasonable agreement.

To test for Ising-like order we define an order parameter analogously to an Ising antiferromagnet,

$$M = \frac{1}{L^2} \sum_i q_i (-1)^{x_i + y_i} . \qquad (11)$$

We first consider $\sigma = 0.3$, smaller than both the naive estimate of $\sigma_c = 0.8$, and the $\sigma_c = \sqrt{\pi/8} \simeq 0.63$ of the $f = 0$ model. Figure 1 plots $[\langle M^2 \rangle]$ vs T, averaged over 200 disorder realizations, for sizes $L = 10, 14$, and 20. All curves start to increase from zero near $T \simeq 0.13$, which is $T_c(\sigma = 0)$ of the pure model. However, $[\langle M^2 \rangle]$ at low T decreases steadily with increasing L. The reason for this becomes clearer if we consider the histogram of values of $\langle M^2 \rangle$ that occur as we sample the different realizations of disorder. We show such histograms in Figs. 1b–1d for the lowest temperature $T = 0.02$. As L increases, the statistical weight shifts from predominantly ordered systems ($M^2 = 1/4$) to predominantly disordered systems ($M^2 = 0$). Assuming that this trend continues, we expect that as $L \rightarrow \infty$, $[\langle M^2 \rangle] \rightarrow 0$.

To measure the "random field correlation length" ξ, we consider the vortex correlation function

$$S(\mathbf{k}) = \frac{1}{L^2} \sum_{i,j} e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle n_i n_j \rangle . \qquad (12)$$

For the pure case, $S(\mathbf{k})$ in the ordered phase has singular Bragg peaks at $\mathbf{K} = \pm \pi \hat{x} \pm \pi \hat{y}$. If the vortex lattice is disordered, these peaks will broaden, and their finite width provides a measure of ξ. Writing $\mathbf{k} = \mathbf{K} + \delta\mathbf{k}$, and assuming a Lorentzian shape for the disorder averaged peak, $[S(\mathbf{k})] \propto 1/(\delta k^2 + \xi^{-2})$, we determine ξ by fitting
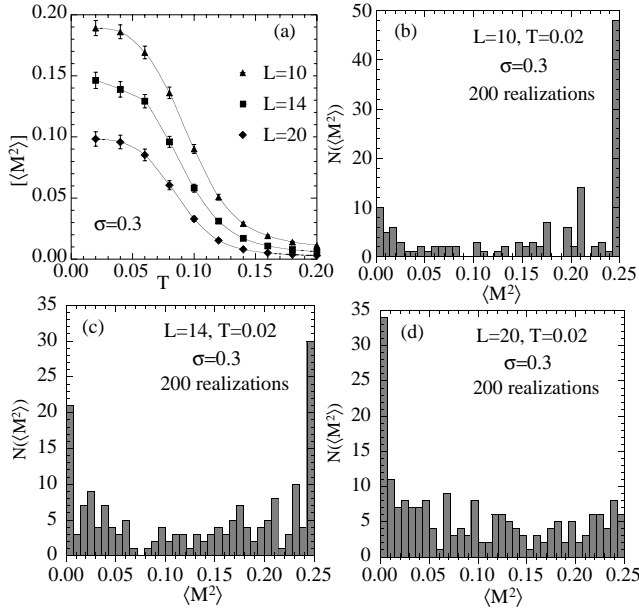
FIG. 1. (a) $[\langle M^2 \rangle]$ vs T, for $\sigma = 0.3$, and sizes $L = 10$, 14, and 20 (solid lines are guides to the eye). Histogram of occurrences of $\langle M^2 \rangle$ in 200 realizations of disorder, for $\sigma = 0.3$ at low $T = 0.02$, for (b) $L = 10$, (c) $L = 14$, and (d) $L = 20$.

to this form for $\delta k = 0$, and $\delta k = 2\pi/L$ [19]. In Fig. 2 we show ξ vs σ at our lowest $T = 0.02$, for several system sizes L. Only for our smallest value $\sigma = 0.25$ does a finite size effect remain. In this case, however, ξ *decreases* as L increases. This is in contrast to the increase of ξ with L that one would expect if one were approaching a second order transition. This behavior is consistent with that seen in Figs. 1b–1d, where as L increases, a greater fraction of the disorder realizations result in disordered states.

We next fit our results for $\xi(\sigma)$ to several possible scaling expressions: (i) $\xi \sim e^{C/(\sigma-\sigma_c)^2}$, (ii) $\xi \sim e^{C/(\sigma-\sigma_c)^{4/3}}$, (iii) $\xi \sim e^{C/(\sigma-\sigma_c)}$, and (iv) $\xi \sim |\sigma - \sigma_c|^{-p}$. The first two, with $\sigma_c \equiv 0$, have been suggested [16] for the 2D RFIM; here we leave σ_c as an arbitrary parameter to be determined from the fit. The third has been suggested for the positionally disordered $f = 0$ model [1,2], in which
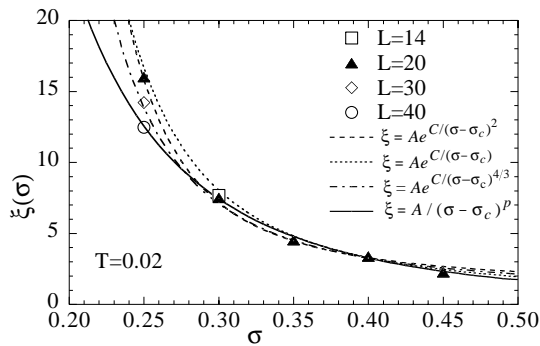

FIG. 2. Correlation length $\xi(\sigma)$ vs σ at $T = 0.02$ for various sizes L. Sizes $L = 14, 20$ are averaged over 200 realizations of the randomness; sizes $L = 30, 40$ are averaged over 50 realizations. Lines are fits to the scaling forms (i)–(iv).

$\sigma_c > 0$. The fourth is the familiar power law form. Using data for only the largest L for each σ, the results of these fits are shown in Fig. 2. The value of σ_c and the χ^2 of the fit for each case is (i) $\sigma_c = 0.0046 \pm 0.035$, $\chi^2 = 67$; (ii) $\sigma_c = 4.1 \times 10^{-6} \pm 0.041$, $\chi^2 = 31$; (iii) $\sigma_c = 0.0134 \pm 0.055$, $\chi^2 = 67$; (iv) $\sigma_c = 0.0013 \pm 0.098$, $p = 2.86 \pm 0.84$, $\chi^2 = 7.6$. The power law (iv) gives the best fit of all the forms; however, all give $\sigma_c = 0$ within the estimated error. Given the rather limited range of the data, the above fits should be treated with caution. However, they do indicate that the data contain no suggestion of a diverging ξ at a finite σ. Coupled with our Imry-Ma argument, we thus find a consistent picture suggesting that $\sigma_c = 0$ for the $f = 1/2$ 2D *XY* model.

Returning to the case $\sigma = 0.3$, where Ising-like order has been lost, we now consider whether the system may still have a finite temperature "spin glass" transition to a disordered but frozen vortex state. To test for this we measure the self and cross overlaps [20], Q_{self} and Q_{cross},

$$Q_{\text{self}\alpha} = \frac{1}{L^2} \sum_i \langle n_i^{(\alpha)}(t) n_i^{(\alpha)}(t+\tau) \rangle,$$

$$Q_{\text{cross}} = \frac{1}{L^2} \sum_i \langle n_i^{(\alpha)}(t) n_i^{(\beta)}(t) \rangle. \tag{13}$$

α and β index the two independent replicas. For τ sufficiently large we expect $Q_{\text{self}1} = Q_{\text{self}2} = Q_{\text{cross}}$, if the system is well equilibrated. Averaging Eq. (13) over several values of $\tau \geq 2000$ to improve our statistics, we plot $[Q_{\text{self}1}]$, $[Q_{\text{self}2}]$, and $[Q_{\text{cross}}]$ vs T in Fig. 3a. We see that our system is fairly well equilibrated down to the lowest T we study. To test for a spin glass transition, we measure the overlap susceptibility,

$$\chi_Q = L^2 \{ [\langle Q_{\text{cross}}^2 \rangle] - [\langle Q_{\text{cross}} \rangle^2] \}, \tag{14}$$

which we plot vs T in Fig. 3b for various system sizes. The peak in χ_Q near $T \simeq 0.06$ shows no noticeable increase as L increases, thus suggesting that there is no finite temperature spin glass transition.

If the vortices are not frozen, but are free to diffuse, one expects that superconducting phase coherence is also destroyed. To explicitly test this we measure the helicity
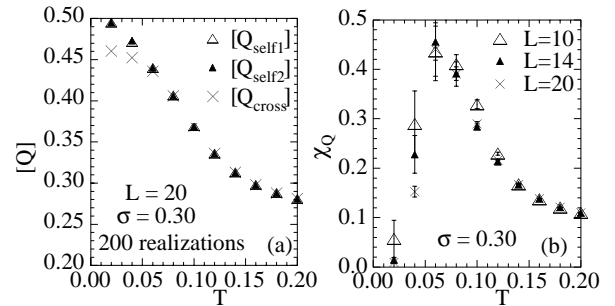

FIG. 3. (a) Overlaps $[Q_{\text{self}1}]$, $[Q_{\text{self}2}]$, and $[Q_{\text{cross}}]$ vs T for $\sigma = 0.3$ and $L = 20$; (b) overlap susceptibility χ_Q vs T for $\sigma = 0.3$ and $L = 10$, 14, and 20. Both are averaged over 200 disorder realizations.

modulus. The Hamiltonian (3) can viewed as representing the XY model with "fluctuating twist" boundary conditions [11]. Using the method of Ref. [21], we determine the dependence of the total free energy F of the corresponding XY model, as a function of the twist (Δ_x, Δ_y) which is applied in a "fixed twist" boundary condition. We then determine the $(\Delta_{x0}, \Delta_{y0})$ that minimizes F; the helicity modulus tensor is then the curvature of F at the minimizing twist, $\Upsilon_{\mu\nu} = \partial^2 F / \partial \Delta_\mu \partial \Delta_\nu$. In Fig. 4a we plot Υ_1, the largest of the two eigenvalues of $\Upsilon_{\mu\nu}$, vs T, for $\sigma = 0.3$ and sizes $L = 10, 14,$ and 20. At all T, Υ_1 continues to decrease as L increases, giving no suggestion of a finite temperature transition. In Figs. 4b–4d we plot histograms of the minimizing twist Δ_0 for the three sizes L. Note in choosing our random phase shifts $\delta A_{i\mu}$, we impose the constraint $\sum_i \delta A_{i\mu} = 0$ in order to remove one trivial source of $\Delta_0 \neq 0$. We see that the width of the distributions of Δ_0 steadily increases with increasing L, suggesting [4] that the strength of the random disorder is renormalizing to greater values on larger length scales.

To conclude, our results suggest that Ising-like order is destroyed for any finite amount of positional disorder. Further, we found in one specific case that when the Ising-like order vanished, no spin glass order or phase coherence existed either. We speculate that this remains true as well for any finite disorder strength. Although $\sigma_c = 0$, the finite $\xi(\sigma)$ nevertheless can become extremely large for small values of σ. When ξ exceeds the size of the experimental or numerical sample, the system will indeed look ordered. We believe this explains previous numerical work on this problem which reported the persistence of Ising-like order at small σ. In the most recent of these works, Cataudel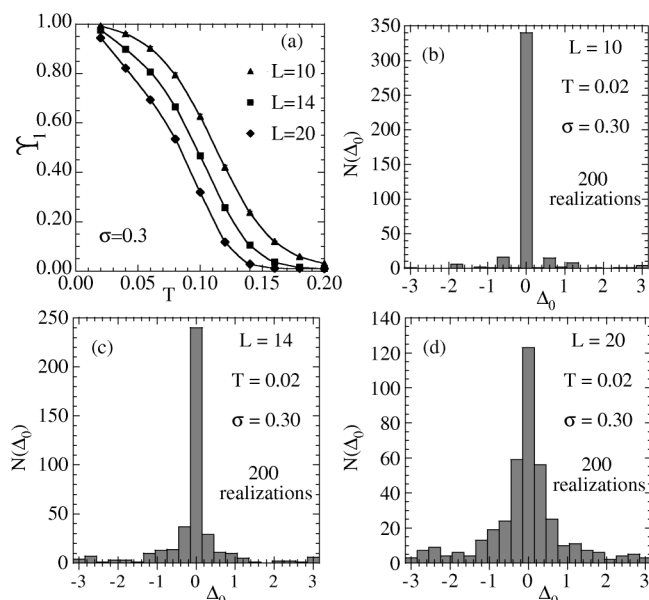la [8] reports at $\sigma \simeq 0.113$ a finite T_c to an Ising-like ordered state. The correlation length exponent that he finds is $\nu \sim 1.7$, clearly different from that of the pure model. Using our scaling form (iv) we can estimate that at this value of σ, $\xi \sim 120$, much larger than Cataudella's largest system size of $L = 36$. His results may thus be reflecting a cross over region at $L < \xi$, rather than a true transition.

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[1] E. Granato and J. M. Kosterlitz, Phys. Rev. B **33**, 6533 (1986); M. Rubinstein, B. Shraiman and D. R. Nelson, Phys. Rev. B **27**, 1800 (1983).

[2] T. Nattermann, S. Scheidl, S. E. Korshunov, and M. S. Li, J. Phys. I (France) **5**, 565 (1995); L.-H. Tang, Phys. Rev. B **54**, 3350 (1996); S. Scheidl, Phys. Rev. B **55**, 457 (1997); M. C. Cha and H. Fertig, Phys. Rev. Lett. **74**, 4867 (1995); J. Maucourt and D. R. Grempel, Phys. Rev. B **56**, 2572 (1997); Y. Ozeki and H. Nishimori, J. Phys. A **26**, 3399 (1993).

[3] M. G. Forrester, S. P. Benz, and C. J. Lobb, Phys. Rev. B **41**, 8749 (1990); A. Chakrabarti and C. Dasgupta, Phys. Rev. B **37**, 7557 (1988); S. P. Benz, M. G. Forrester, M. Tinkham, and C. J. Lobb, Phys. Rev. B **38**, 2869 (1988); M. G. Forrester, H. J. Lee, M. Tinkham, and C. J. Lobb, Phys. Rev. B **37**, 5966 (1988); S. E. Korshunov, Phys. Rev. B **48**, 1124 (1993).

[4] J. M. Kosterlitz and M. Simkin, Phys. Rev. Lett. **79**, 1098 (1997).

[5] J. Villain, J. Phys. C **10**, 1717 (1977); **10**, 4793 (1977).

[6] S. Teitel and C. Jayaprakash, Phys. Rev. B **27**, 598 (1983); Phys. Rev. Lett. **51**, 1999 (1983).

[7] E. Granato and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 823 (1989); M. Y. Choi, J. S. Chung, and D. Stroud, Phys. Rev. B **35**, 1669 (1987).

[8] V. Cataudella, Europhys. Lett. **44**, 478 (1998).

[9] J. Villain, J. Phys. (Paris) **36**, 581 (1975).

[10] E. Fradkin, B. Huberman, and S. H. Shenker, Phys. Rev. B **18**, 4789 (1978); A. Vallat and H. Beck, Phys. Rev. B **50**, 4015 (1994).

[11] P. Olsson, Phys. Rev. B **52**, 4511 (1995).

[12] Y. Imry and S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).

[13] C. Denniston and C. Tang, Phys. Rev. Lett. **79**, 451 (1997); Phys. Rev. B **58**, 6591 (1998).

[14] T. C. Halsey, J. Phys. C **18**, 2437 (1985).

[15] J.-R. Lee and S. Teitel, Phys. Rev. B **46**, 3247 (1992).

[16] K. Binder, Z. Phys. B **50**, 343 (1983).

[17] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989); K. Hui and N. Berker, *ibid.* **62**, 2507 (1989).

[18] K. A. Benedict and M. A. Moore, Phys. Rev. B **39**, 4592 (1989).

[19] P. Olsson, Phys. Rev. B **55**, 3585 (1997).

[20] R. N. Bhatt and A. P. Young, Phys. Rev. B **37**, 5606 (1988).

[21] P. Gupta, S. Teitel, and M. J. P. Gingras, Phys. Rev. Lett. **80**, 105 (1998).

FIG. 4. (a) Υ_1 vs T, for $\sigma = 0.3$, and sizes $L = 10, 14,$ and 20 (solid lines are guides to the eye). Histogram of values of Δ_0 found in 200 disorder realizations, for $\sigma = 0.3$ at low $T = 0.02$, at (b) $L = 10$, (c) $L = 14$, and (d) $L = 20$.