Equilibrium Phase Transitions in Josephson Junction Arrays

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Abstract. I review several problems dealing with the equilibrium behavior of classical two dimensional Josephson junction arrays in applied magnetic fields. Specific attention is given to the cases of a uniform field with average flux density per unit cell of $f = 0$, $f = 1/2$, $f = 1/q$ and $f = 1/2 - 1/q$. Several models incorporating the effects of randomness on the Josephson array are also reviewed. These include the case of a random vortex pinning potential and its effects on vortex lattice order, and the spin glass, gauge glass, and positionally disordered array.

1 Introduction

Two dimensional arrays of coupled Josephson junctions have been a topic of much experimental and theoretical investigation for approximately the last fifteen years. As they are a system in which topological singularities, frustration, incommensurability, and randomness can all come into play, they serve as an excellent model for studying many diverse problems in statistical physics. More recently, Josephson arrays have received attention in connection with high temperature superconductors. Both are superconducting systems in which thermal fluctuations play a crucial role in determining the macroscopic behavior. The Josephson array, with its simpler phase space and well defined geometry, can therefore serve as a test case for understanding many issues of importance to high T_c superconductors, such as vortex pinning and the effects of randomness.

Yet despite nearly fifteen years of investigation, there remain fundamental unresolved issues in even some of the most simply posed problems. In this article I will review some of the theoretical work concerning the equilibrium phase transitions in classical Josephson arrays, placing emphasis on results from numerical simulations. I will try to point out what is understood, and what questions remain open.

1.1 The Josephson Junction Array Model

The system of the Josephson array can be conceptualized as follows. At each site of a grid of points there is a superconducting island. Nearest neighbor islands are coupled to each other by the tunneling of Cooper pairs (either though proximity effect barriers, or oxide layer barriers), thus producing a

Josephson junction on each bond of the grid. The relevant degrees of freedom of the system are then the phase angles θ_i of the superconducting wavefunction on sites i of the grid. If the array is placed in an external magnetic field given by the vector potential $\mathbf{A}$, one defines on nearest neighbor bonds $\langle ij\rangle$ the integrals $A_{ij} = (2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot d\boldsymbol{\ell}$, where $\Phi_0 = 2e/hc$ is the flux quantum. The Hamiltonian of the array is then just the sum of Josephson energies for each nearest neighbor bond,

$$\mathcal{H}[\theta_i] = \sum_{\langle ij\rangle} V_{ij}(\theta_i - \theta_j - A_{ij}) \ , \tag{1}$$

where $V_{ij}(\phi)$ is the coupling energy of bond $\langle ij\rangle$, and its argument is the gauge invariant phase angle difference across the bond. $V_{ij}(\phi)$ is quadratic about its minimum at $\phi = 0$, has period 2π, and $dV_{ij}/d\phi$ is proportional to the supercurrent flowing through the bond. I consider here only the classical case in which charging energy, and hence quantum effects, can be ignored. The equilibrium behavior of the array is then obtained from the partition function, summing $e^{-\beta\mathcal{H}}$ ($\beta = 1/k_B T$) over all possible configurations of the phases $\{\theta_i\}$. If one ignores inductance effects, the A_{ij} remain fixed parameters determined from the applied magnetic field. Here I will consider only the case where the grid of sites i form a periodic two dimensional lattice, which unless stated otherwise, I take to be square. I will also assume that there is no randomness in the couplings, and hence the $V_{ij}(\phi) \equiv V_0(\phi)$ are all equal. Periodic boundary conditions will be imposed at the edges of the system.

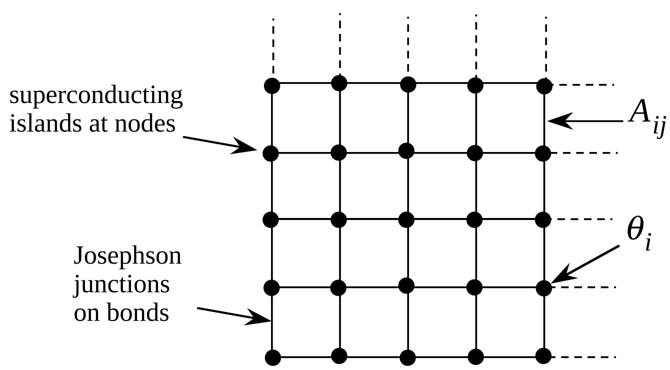

Fig. 1. Schematic geometry of a Josephson junction array

The different junctions of the array are coupled to each other through a topological constraint on the θ_i. Since θ_i is the phase angle of a complex wavefunction, one must find that the sum of phase angle differences $[\theta_i - \theta_j] \in (-\pi, \pi]$, going around any closed path on the grid must sum to $2\pi n$, with n

integer. n is the net vorticity contained within the path. Such vortices are the key excitations of the Josephson array. In the absence of any applied magnetic field, $A_{ij}=0$, and the ground state has all θ_i equal, so that $V_0(\phi)$ is minimized on each bond. When $A_{ij}\neq 0$, the above topological constraint will in general prevent the θ_i from adjusting so as to cancel out the effect of the A_{ij}. The result will be a ground state in which the θ_i have some spatially varying pattern, $V_0(\phi)$ is no longer minimized on all bonds, and so there is a flow of finite local supercurrents. Since each bond is now no longer able to achieve its minimum energy, such a model is said to be *frustrated.* For the case where the A_{ij} describe a uniform magnetic field perpendicular to the plane of the array, (1) is referred to as the *uniformly frustrated* 2D XY model.

1.2 Coulomb Gas Duality

For a Josephson junction one typically uses $V_0(\phi)=-J_0\cos(\phi)$, where the coupling constant J_0 is related to the critical current of the junction I_0 by, $I_0=(2e/\hbar)J_0$. However much theoretical simplification can be achieved by using instead the Villain function (Villain 1975), defined by

$$\mathrm{e}^{-\beta V_0(\phi)} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\beta J_0(\phi-2\pi m)^2} \quad . \tag{2}$$

In this case, exact duality transformations (José et al. 1977, Fradkin et al. 1978, Vallat and Beck 1994) map (1) onto an equivalent problem of logarithmically interacting charges, given by the Hamiltonian,

$$\mathcal{H}[n_i] = \tfrac{1}{2}e^2\sum_{i,j}(n_i-f_i)G(\mathbf{r}_i-\mathbf{r}_j)(n_j-f_j) \quad , \tag{3}$$

Here the *dual* sites i and j sit at the centers of the unit cells of the original grid; f_i is the sum of the $A_{ij}/2\pi$ going counterclockwise around the unit cell at i and gives the number of flux quanta of applied field penetrating cell i; $n_i=0,\pm1,\pm2,\ldots$ are integers; the unit of charge is $e=\sqrt{2\pi J_0}$; and $G(\mathbf{r})$ is the solution to the lattice Laplacian with periodic boundary conditions, which can be explicitly written for a square lattice of length L as,

$$G(\mathbf{r}) = \frac{2\pi}{L^2}\sum_{k\neq 0}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{4-2\cos k_x-2\cos k_y} \quad . \tag{4}$$

Here the sum is over all wavevectors satisfying periodic boundary conditions, $k_\mu=\frac{2\pi}{L}m_\mu$, $m_\mu=0,1,\ldots,L-1$, and the grid spacing has been taken to be unity. For large separations, $1\ll r\ll L$, $G(\mathbf{r})\simeq-\ln r$.

The Hamiltonian (3) can be viewed as one of interacting integer charges n_i, superimposed on a quenched background charge distribution f_i, and is referred to as the 2D *Coulomb gas* (CG). The partition function is obtained summing over all charge configurations $\{n_i\}$ subject to the constraint of total charge neutrality $\sum_i (n_i - f_i) = 0$. In this duality mapping, the integer charges n_i are related to the vortices in the phase angles θ_i of the original Josephson array (Vallat and Beck 1994). Smooth spin wave fluctuations of the θ_i about a given vortex configuration give an additional contribution to the system energy, which is decoupled from the vortex part (3). This spin wave part is Gaussian, and so may be directly summed in the partition function, giving a non-singular additive contribution to the free energy which is henceforth ignored.[1] This Coulomb gas formulation of the Josephson array will form the basis for most of the following discussion. Henceforth, I will use the terms "charge" and "vortex" interchangeably.

The partition function derived from (3) is invariant under the transformation $f_i \to f_i + m_i$, where m_i is any integer, as such a change in f_i can always be canceled out by the addition of an appropriate integer charge n_i to site i. It is thus sufficient to consider only cases with $-\frac{1}{2} < f_i \leq \frac{1}{2}$.

1.3 The Kosterlitz-Thouless Instability Criterion

In terms of the Coulomb gas model (3) one can search for an insulator to conductor transition by considering the inverse dielectric function ϵ^{-1}. Defining the net charge on site i by $q_i \equiv n_i - f_i$, and its Fourier transform by $q_k = \sum_i \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} q_i$, linear response theory gives (Minnhagen 1987),

$$\epsilon^{-1}(T) = \lim_{k\to 0} 1 - \frac{2\pi e^2}{L^2 T} \frac{\langle q_k q_{-k}\rangle}{k^2} \quad . \tag{5}$$

The conducting state is one in which there are freely diffusing charges and $\epsilon^{-1} = 0$. In the insulating state, charges n_i are bound either to each other in neutral clusters, or to the background f_i, and $\epsilon^{-1} > 0$. Since a charge is identified with a vortex in the Josephson array, and each time a vortex crosses a path the total phase angle difference along that path changes by 2π, the free diffusion of charges (vortices) will correspond to large phase angle fluctuations that destroy superconducting coherence. Thus one can show that the conducting (insulating) phase of the Coulomb gas corresponds exactly to the normal (superconducting) phase of the Josephson array (Ohta and Jasnow 1979, Minnhagen 1987).

A criteria for predicting the instability of the insulating phase has been given in the pioneering work of Kosterlitz and Thouless (1973) (KT). Consider the energy U of a single *free* charge e, including the effect of the dielectric screening due to other *bound* charges. Since the dominant effects come from large length scales, one can make a continuum approximation, giving $U = \int d^2r |\mathbf{E}(\mathbf{r})|^2/(4\pi\epsilon) = (e^2/2\epsilon)\ln(L/a)$, where $\mathbf{E} = e/|\mathbf{r}|$ is the electric field of

[1] Note however that a coupling between spin waves and vortices remains in the original XY model (1) if the cosine interaction is used in place of the Villain interaction (Ohta and Jasnow 1979).

the charge e, L is the radius of the system, and a is the hard core radius of the charge. The entropy of the free charge is just the logarithm of the number of non-overlapping positions in which the charge may be placed, $S = \ln(L/a)^2$. Thus the total free energy for a free charge to appear is,

$$F = U - TS = \left(\frac{e^2}{2\epsilon(T)} - 2T\right)\ln(L/a) \ . \tag{6}$$

As $L \to \infty$, $F \to \pm\infty$, depending of the sign of the prefactor of the logarithm. Thus when $T < e^2/4\epsilon$, it costs infinite free energy to create an isolated free charge, and the insulating phase is stable against such an excitation. When $T > e^2/4\epsilon$, however, the free energy for a free charge is infinite but negative; free charges proliferate and the insulating phase becomes unstable. If T_c is the true insulator to conductor transition temperature, one then has the following bound,

$$T_c \leq e^2/4\epsilon(T_c) \ . \tag{7}$$

The above gives only an upper bound on T_c, as it is always possible that an excitation more complicated than the isolated free charge considered above, may drive the insulator to conductor transition.

The above bound on T_c also gives a very important result concerning the behavior of the dielectric function. One can rewrite (7) as,

$$\epsilon^{-1}(T_c) \geq 4T_c/e^2 \ . \tag{8}$$

Since one has $\epsilon^{-1} = 0$ for $T > T_c$ in the conducting state, (8) implies that for finite T_c, ϵ^{-1} must make a discontinuous jump to zero at the transition.

2 Uniform Frustration

In this section I focus on behavior for the case of a Josephson array in a uniform applied magnetic field (Teitel and Jayaprakash 1983a, 1983b). This corresponds to the CG with a uniform quenched background charge, i.e. all $f_i = f$, a constant. By the condition of charge neutrality, the ground state will consist of a finite density of integer charges (vortices) with $\sum_i n_i = fL^2$. For rational $f = p/q$ these ground state charges should be arranged in a periodic structure. However finding this ground state structure for a general $f = p/q$ remains an unsolved problem. The competition between the repulsive charge-charge interaction and the grid geometry of allowed charge sites, leads to complicated commensurability effects and discontinuous behavior as f is smoothly varied. A particularly clear experimental verification of such commmensurability effects has been seen in kinetic inductance measurements on a triangular Josephson array (Théron et al. 1994).

To find ground states in specific cases, one must resort to symmetry arguments combined with a numerical search through likely candidate states. The most extensive listing of ground states, considering all $f = p/q$ for $q \leq 20$, has

Fig. 2. Ground state charge (vortex) configurations for various values of $f = 1/q$ and $f = 1/2 - 1/q$. (•) denotes a charge; for $f = 5/11$, a shaded box denotes a charge missing from the 1/2–like background.

been given by Straley and Barnett (1993). In Fig. 2 are shown some selected ground states for two special cases: $f = 1/q$ and $f = 1/2 - 1/q$.

For $f = 0$, the ground state is a charge vacuum. For $f = 1/2$, the ground state is a checkerboard pattern of charges, with a double discrete degeneracy corresponding to the two possible sublattices which the integer charges $n_i = +1$ may occupy. For $f = 1/q$, the ground state is a periodic lattice that, as q gets larger, becomes an increasingly better approximation to the triangular lattice that would be found for vortices in a continuum. Note that for $f = 1/4$ there are two different degenerate configurations (Korshunov 1986, Straley and Barnett 1993). For $f = 1/2 - 1/q$, the ground state looks almost everywhere like the checkerboard pattern of $f = 1/2$ except for localized defect regions, which are required so as to give the correct charge density $f < 1/2$. For small values of q these defect regions take the form of domain walls between the two degenerate $f = 1/2$–like ground states; these are indicated by the heavy lines in Fig. 2 for $f = 2/5$ and $3/7$. For larger

q, the defects take the form of a supperlattice of missing charges in a single $f = 1/2$–like ground state; these are indicated by the shaded boxes in Fig. 2 for $f = 5/11$. Note that the ground state for $f = p/q$ is often described by a $q \times q$ unit cell. However this is not generally true. As shown in Fig. 2 for example, $f = 5/11$ is periodic with a $2q \times 2q$ unit cell.

How the ground state charge (vortex) structure melts upon increasing temperature is another question for which the general answer remains unknown. I will consider in greater detail below the cases $f = 0$, $f = 1/2$, $f = 1/q$ and $f = 1/2 - 1/q$ for *large* q. For $f = 1/5$ and $f = 2/5$ it is known that the charge lattice has a first order melting transition to a conducting liquid (Li and Teitel 1990, 1991). The simple fractions $f = 1/3$ and $1/4$ have received some study (Grest 1989, Lee and Lee 1995), however the critical behavior of these transitions remains poorly understood.

2.1 Ordinary XY Model: $f = 0$

The case $f = 0$ corresponds to the ordinary two dimensional XY model, originally studied by Kosterlitz and Thouless. The ground state is the vacuum and the low temperature excitations of the insulating phase consist of dipoles formed of bound $n_i = +1$, $n_j = -1$, charge pairs. As T increases, the average separation of the charges in these dipoles increases, until at a critical temperature T_c, the charge pairs unbind giving free charges and a conducting phase. The renormalization group (RG) recursion equations by Kosterlitz (1974) quantify this picture and yield the result that the KT instability criterion (7) is satisfied as an exact equality. Equivalently, the discontinuous jump to zero of $\epsilon^{-1}(T_c)$ has the *universal* value $4T_c/e^2$. Although ϵ^{-1} is discontinuous, the transition is second order with an infinite correlation length at T_c.

This KT transition has been well established both experimentally, as well as by high precision Monte Carlo simulations (Olsson 1995a). For a good theoretical and experimental review see Minnhagen (1987). However the Kosterlitz RG analysis is an expansion in charge fugacity that applies only at small average charge densities, $\rho \equiv L^{-2}\sum_i |n_i|$. At large ρ one can question if the KT mechanism continues to hold. Several authors, using continuum models, have attempted to extend the KT analysis to higher charge densities by including the screening effect of free charges on bound charges (Minnhagen and Wallin 1987, 1989, Thijssen and Knops 1988b, Levin et al. 1994, Friesen 1995). They have predicted that the KT transition becomes first order as the fugacity increases. In particular, Levin at al. predict that the KT transition becomes first order at the relatively low charge density of $\rho_c = 0.0039/a^2$, where a is the diameter of the hard core charge.

To investigate numerically the behavior of the $f = 0$ CG at large charge densities, one can add a chemical potential term to the Hamiltonian (3),

$$\mathcal{H}[n_i] = \tfrac{1}{2}\sum_{i,j} n_i G(\mathbf{r}_i - \mathbf{r}_j) n_j - u\sum_i n_i^2 + \sum_i (n_i^4 - n_i^2) \ . \tag{9}$$

Fig. 3. Phase diagram of the dense $f = 0$ CG

There is indeed a first order transition (heavy solid line), however it separates the insulating gas of charge dipoles from an *insulating* charge solid of alternating $+1$ and -1 charges. As this charge solid is heated, there is first a KT transition to a conducting solid in which charge interstitials and vacancies can diffuse freely, followed by an Ising–like melting of the solid to a liquid. The two KT transition lines, in the solid phase and in the dilute gas phase, meet the first order line at the same temperature, $T^* \simeq 0.126$, somewhat below the tricritical point where the Ising melting line and the first order line meet.

The origin of the charge solid phase is easy to see by considering the Fourier transform of the first two terms of (9), $\mathcal{H} = L^{-2} \sum_k \left(\frac{1}{2} G_k - u\right) n_k n_{-k}$. As u increases, the ground state will change from the vacuum to an ordered charge solid when $2u$ first equals the smallest value of G_k. For the square grid Green's function (4) this occurs for $\mathbf{k} = \pi\hat{x} + \pi\hat{y}$ (thus giving the checkerboard pattern of the ground state) at $u_0 = \pi/8$.

As the first order line in Fig. 3 is a direct consequence of the formation of a charge solid, it is presumably unrelated to the first order transitions predicted by the above theoretical works, which are all built around charge pairs as the fundamental charge correlation.[2] It is thus worthwhile to give closer attention to the transition line from the gas of dipoles to the liquid, which is labeled as "KT" in Fig. 3. To test that this line does indeed remain a second order phase transition all the way up to the first order line at $u = u_0$, Gupta and Teitel (1996) have computed histograms of charge density, as temperature is varied for fixed $u = 0.39$ just below the first order line at $u_0 = \pi/8 \simeq 0.3937$. If the transition is first order, there should be a discontinuity in charge density as the first order line is crossed. One thus expects, in the vicinity of the transition, to see a charge density histogram with two distinct peaks corresponding to the two differing average charge densities of the two phases. If however the transition is second order, with no discontinuity in charge density, only a single peak should be present. The numerically computed histograms are shown in Fig. 4 below, for a system of size $L = 64$. As is seen, there is no hint at all of a bimodal distribution for any of the temperatures in the vicinity of the transition. While one can not rule out the possibility of a very weak first order transition, with finite correlation length $\xi(T_c) \gg L = 64$, the present evidence suggests that the transition remains second order, and presumably remains in the KT universality class. One can also see from Fig. 4 that the average charge density at this KT transition is $\rho \simeq 0.11$, well above the ρ_c estimated by Levin et al.

The phase diagram of Fig. 3, and in particular the presence of the charge solid phase, is very strongly influenced by the fact that the CG has been placed on a square grid of allowed charge sites. If a different geometry for the grid is used, the locations of the various phase boundaries shift, and charge solids with different symmetries can form (Lee and Teitel 1992). One can speculate whether for such another grid, or for charges in a continuum, it is possible that the melting transition line of the charge solid moves down to lower temperatures, so that it intersects the first order line at a temperature below T^*. One then could have a first order transition from an insulating gas of dipoles (or more complicated neutral clusters) to a very dense conducting liquid. Such a result has been reported by Caillol and Levesque (1986), who simulate a hard core CG in a continuum, placing the charges on the two dimensional surface of a sphere. Their KT line ends at a temperature and

[2] Nevertheless, it is interesting to note that the values of $u_0 = \pi/8$ and $T^* = 0.126$ in Fig. 3 do in fact lie fairly close to the values predicted by Minnhagen and Wallin. In order to compare the u_0 for the first order line of the CG on the square grid with Minnhagen and Wallin's results for a continuum CG, it is necessary to note (Kosterlitz and Thouless 1973) that if the interaction $G(\mathbf{r})$ on the grid is chosen so as to asymptotically match the continuum $-\ln r$ as $r \to \infty$, then the grid CG with $u = 0$ acts like a continuum model with a chemical potential $\mu_0 = -\frac{1}{2}(\gamma + \frac{3}{2}\ln 2) \simeq 0.8085$, where $\gamma \simeq 0.5772$ is Euler's constant. Thus a chemical potential $u_0 = \pi/8$ on the grid acts like a chemical potential $\mu = u_0 - \mu_0 = -0.416$ in the continuum. Minnhagen and Wallin predict that the KT line will end at $T^* = 0.144$, and chemical potential $\mu = T^* \ln z^* = -0.420$.

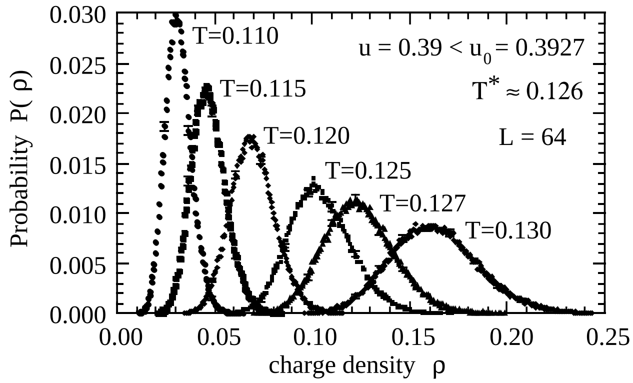

Fig. 4. Charge density histograms at various T, as the KT transition line is crossed for $u = 0.39$ just below $u_0 = \pi/8$

density comparable to the values found on the square grid, only they report no solid phase above the first order line. However, while the discrete square grid explicitly favors the formation of a charge solid, the surface of a sphere explicitly discourages it. In order to fit a periodic solid to the curvature of the spherical surface, it is necessary to introduce lattice defects, which might remain mobile even at low temperatures (Dodgson 1995).

Most recently, Lidmar and Wallin (1996) have carried out simulations of the hard core CG in a flat continuum with periodic boundary conditions. They find that the KT line remains second order down to quite low temperature and high density, until it finally hits a first order line at $T^* \simeq 0.032$. As in the case of the discrete grid, this first order line is associated with a transition in ground state from the vacuum to a charge solid. However unlike the discrete grid, their evidence suggests that the charge solid is melted at any finite temperature. Their first order line is thus a transition from an insulating gas to a dense conducting liquid.

2.2 Fully Frustrated Case: $f = 1/2$

Square Grid: On a square grid, the $f = 1/2$, or fully frustrated, CG consists of the checkerboard ground state shown in Fig. 2. This ground state breaks two distinct symmetries: (i) the continuous symmetry associated with uniform rotation of all phase angles θ_i of the original Josephson array model (1) - in the CG this is reflected in the insulating nature of the ground state; and (ii) the discrete translational symmetry broken by choosing one of the two equivalent sublattices on which to place the integer charges n_i. A natural question is whether these two symmetries are broken at one single, or two distinct, transition temperature(s).

Fig. 5. Two types of excitations of the $f = 1/2$ ground state

The second type of excitation, referred to as "Ising–like," consists of reversing the sign of all charges within a neutral domain. For domains which contain no net dipole moment, the energy of such an excitation is proportional to the perimeter of the domain. Such domain excitations, which can involve large numbers of charges, can not be accounted for within the small fugacity expansion of the KT analysis. It is these domain excitations that will melt the charge solid and restore the translational symmetry (*ii*). Since the ground state is doubly degenerate, one would naively expect this melting to be in the Ising universality class. Once the charge lattice has melted into a charge liquid, with freely diffusing charges, one expects $\epsilon^{-1} = 0$.

There are thus two likely scenarios: (*i*) Upon heating, the KT–like excitations cause a KT transition at $T_{\rm KT}$ with a universal jump in $\epsilon^{-1}(T_{\rm KT})$. This is followed at a higher $T_{\rm I}$ by the melting of the charge solid with Ising critical exponents. Or, (*ii*) the two types of excitations become sufficiently coupled that there is only a single transition T_c. In such a case the jump in $\epsilon^{-1}(T_c)$ might be larger than the universal value, and the melting could perhaps even be in a different universality class than Ising.

Early numerical works (Teitel and Jayaprakash 1983a, Miyashita and Shiba 1984, Lee et al. 1986, Thijssen and Knops 1988a, Grest 1989, Nicolaides 1991) focused, with inconclusive results, on determining whether there was one single, or two separate, transition(s). More recent works (Lee et al. 1991, Granato and Nightingale 1993, Ramirez-Santiago and José 1994, Lee 1994, Lee and Lee 1994, Knops et al. 1994) have focused on finite size scaling analyses of the transitions. They have reported critical exponents for melting distinctly different from Ising values, and jumps in ϵ^{-1} larger than the universal KT value, thus suggesting scenario (*ii*). Recently, Nightingale et al. (1995) have carried out Monte Carlo transfer matrix simulations on a related coupled XY-Ising model believed to be in the same universality class as the $f = 1/2$ CG. Considering systems of size $L \times \infty$, $L \leq 30$, they similarly find non-Ising melting and a larger than universal jump in ϵ^{-1}. However they find that the *central charge* (or conformal anomaly number) has not yet converged to its asymptotic large L limit, and the XY degrees of freedom similarly seem to show large corrections to scaling. They say that this " ... calls into question the validity of the basic assumption of scaling theory, viz., that there is a single divergent length scale in this system as the critical point is approached..."

Most recently Olsson (1995b), using system sizes up to 128×128, has presented evidence supporting scenario (*i*). According to his arguments, the correlation length $\xi_{\rm KT}(T)$ of the insulator to conductor transition, which diverges at $T_{\rm KT}$, is still very large at the slightly higher melting transition $T_{\rm I}$. This additional length scale at $T_{\rm I}$ complicates the finite size scaling analysis of the melting transition. Only for system sizes $L \gg \xi_{\rm KT}(T_{\rm I})$ will one find a simple scaling characterized by Ising critical exponents. The non-Ising values found in previous works, according to his argument, merely reflect too small values of L. A conclusive resolution of the nature of the transition(s) in this model thus apparently awaits simulation of larger size systems.

Similar questions remain for other cases where the simple symmetry of the ground state charge lattice would naively suggest a melting transition in a well known universality class. One example is the $f = 1/3$ CG on a triangular grid, which might be expected to be in the 3–state Potts universality class (Korshunov 1986, Lee and Teitel 1992).

Triangular Grid: Another interesting model is the $f = 1/2$ CG on a triangular grid of sites (corresponding to a honeycomb Josephson array) (Korshunov 1986). For the triangular grid, the Green's function giving the charge interaction is given in terms of its Fourier transform by (Lee and Teitel 1992),

$$G_k = \frac{3\pi}{6 - 2\cos(\mathbf{k}\cdot\mathbf{a}_1) - 2\cos(\mathbf{k}\cdot\mathbf{a}_2) - 2\cos(\mathbf{k}\cdot\mathbf{a}_3)} \ , \qquad (10)$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the two basis vectors of the triangular grid, and $\mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1$.

A natural candidate for the ground state would have alternating $\pm\frac{1}{2}$ charges along one of the grid directions $\mathbf{a}_\mu$ ($\mu = 1,2,3$) and be uniform in the remaining directions. Such a state is characterized by a wavevector with $\mathbf{k}\cdot\mathbf{a}_\mu = \pi$, but $\mathbf{k}\cdot\mathbf{a}_\nu = 0$, $\nu \neq \mu$. However, looking at the form (10) for G_k, one can easily see that all wavevectors such that $\mathbf{k}\cdot\mathbf{a}_\mu = \pi$ have degenerate values of G_k, regardless of their components along the other directions $\mathbf{a}_\nu$. The ground state is thus a configuration in which the charges oscillate in sign in one of the three directions $\mathbf{a}_\mu$, but are completely random in the other directions. The degeneracy[3] of the ground state is thus 3×2^L. An example of one such a ground state, which oscillates in the $\mathbf{a}_1$ direction, is shown in Fig. 6 below.

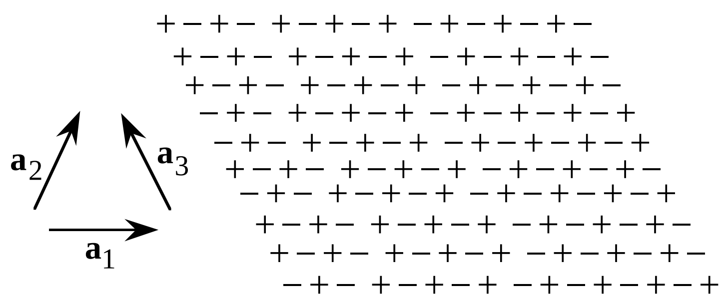

Fig. 6. $f = 1/2$ on a triangular lattice: one of the 3×2^L possible ground states

Such a large ground state degeneracy makes this system a good candidate for glassy behavior. However it remains to ask whether the system can have a true finite temperature equilibrium glass transition, or whether $T_c \to 0$ and so any glassiness would be a result of non-equilibrium effects. Lee and Teitel (1992) have carried out Monte Carlo simulations of this model, computing the dielectric function ϵ^{-1} and specific heat per site C. The results are shown in Fig. 7 below.

One sees in ϵ^{-1} an apparent transition at finite $T_c \simeq 0.035$. The non diverging peak and steep drop of C at T_c are characteristic of a structural glass

[3] Korshunov (1986) argues that the 2^L degeneracy may be lifted in the original XY model (1) with a cosine interaction, due to the coupling between spin waves and vortices that the cosine introduces.

Fig. 7. $f = 1/2$ on a triangular lattice: $\epsilon^{-1}(T)$ and specific heat $C(T)$ for various system sizes L

transition. However to see whether this behavior reflects a true thermodynamic transition, or is rather a freezing out of equilibrium due to large energy barriers in phase space, consider the excitation of a ground state shown in Fig. 8 below. The excitation is formed by taking a $2 \times \ell$ domain and reversing the sign of all the charges. ℓ is chosen in the same direction as that in which the charges oscillate, and is chosen to be even, so that forming the domain creates no net dipole moment. The excitation energy $\Delta E(\ell)$ vs. ℓ is also plotted in Fig. 8. One sees that for $\ell > 10$, $\Delta E(\ell)$ saturates to a finite value $\Delta E(\infty)$. It is easy to see why this is so. If ℓ is parallel to direction $\mathbf{a}_1$ for example, then since all configurations with any sequence of charges in direction $\mathbf{a}_2$ are degenerate ground states, the domain wall segments parallel to $\mathbf{a}_1$ do not look *locally* like domain walls at all! Locally, they are consistent with the system being in one of the other degenerate ground states. It is only near the domain wall "end" segments parallel to the $\mathbf{a}_3$ direction that one notices one is not in a ground state. As the regions near these "end" segments are charge neutral and have no net dipole moment, any interaction between the two ends decays rapidly with increasing ℓ, and so contributions to $\Delta E(\ell)$ arise solely from the local distortion of the ground state near these ends.

Since $\Delta E(\infty)$ is finite, there is only a finite energy barrier to create infinitely long domains. As the growth of such infinitely long domains causes transitions between the 2^L different ground states which are all oscillating in the same direction, the system must in principle remain disordered among these 2^L states at any finite T. The time to hop between these ground states $\sim \mathrm{e}^{\Delta E(\infty)/T}$, however, will get extremely long at low T.

Although the above domain excitations will disorder the system among the 2^L different ground states that oscillate in the same lattice direction, they will not cause transitions between ground states which oscillate in different directions. It thus remains an open question whether there can be a finite

Fig. 8. $f = 1/2$ on a triangular lattice: domain excitation of ground state of length ℓ and excitation energy $\Delta E(\ell)$

temperature equilibrium transition which orders the system into one of the three possible classes of ground states (i.e. into the class of states in which the charges all oscillate in a particular direction $\mathbf{a}_\mu$), but leaves the system disordered among the 2^L degenerate ground states in that class.

2.3 Dilute Case: $f = 1/q$

Next consider the case of a dilute density of charges, $f = 1/q$. For large q, the ground state will be the closest approximation to a triangular charge lattice that is commensurate with the underlying square grid of allowed sites. The melting of this charge lattice, as $q \to \infty$, becomes a model for the melting of the vortex lattice in a thin superconducting film.

Consider as a example the specific case of $f = 1/51$. The ground state charge configuration is shown in Fig. 9*a* below. This case is chosen because, in contrast to other $f = 1/q$ with large q, where the ground state becomes very closely triangular, here the ground state remains very close to square, and so the influence of the discrete grid is strongest.

A convenient quantity for studying the melting of this charge lattice is the charge density structure function,

$$S(\mathbf{k}) = \frac{1}{fL^2} \sum_{i,j} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \langle n_i n_j \rangle \ , \tag{11}$$

which gives the diffraction pattern that would be obtained from scattering off the charge positions. Together with the inverse dielectric function $\epsilon^{-1}(T)$, $S(\mathbf{k})$ will characterize the phase transitions of the system. Heating from the ground state, Franz and Teitel (1995) have computed $S(\mathbf{k})$ and ϵ^{-1} within Monte Carlo simulations. Shown in Figs. 9*b*-*d* are the resulting intensity plots of $S(\mathbf{k})$, for k_x, $k_y \in (-\pi, \pi]$, at three representative temperatures. $\epsilon^{-1}(T)$ is shown in Fig. 10. At low T, Fig. 9*b*, one sees a periodic lattice of *sharp* Bragg peaks, reflecting the *long range* translational order of the charges which

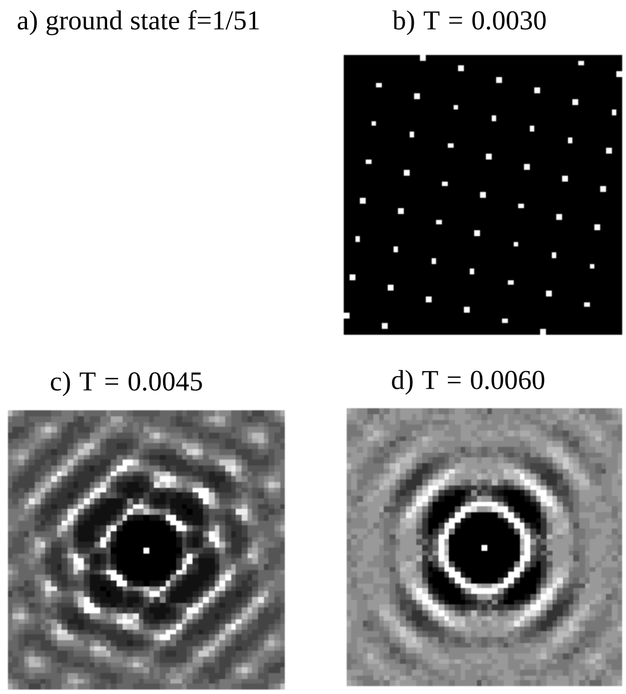

Fig. 9. (a) Ground state charge configuration for $f = 1/51$; (b-d) Structure function $S(\mathbf{k})$ for $T = 0.003$, 0.0045, and 0.006

At an intermediate T, Fig. 9c, one sees a triangular lattice of peaks of *finite width*. This is characteristic of the *quasi-long range* translational order expected for a 2D solid in a uniform continuum (Kosterlitz and Thouless 1973). Note that the peaks in Fig.9c are in distinctly different locations compared to the peaks in Fig.9b. The inverse dielectric function ϵ^{-1} has now vanished indicating that the system is conducting; the charge lattice is free to diffuse as a correlated structure. Thus a transition has occurred from a pinned commensurate almost-square charge lattice, to a floating incommensurate triangular charge lattice. Although the translational coupling to the underlying grid has been destroyed, there remains a strong orientational coupling. The minimum energy corresponds to the case where one of the three basis directions of the triangular charge reciprocal lattice aligns with one of

Fig. 10. Inverse dielectric function $\epsilon^{-1}(T)$ for $f = 1/51$

the two diagonal directions of the square grid. This results in two possible distinct orientations for the floating charge lattice (one of which is clearly seen in Fig. 9*c*) which break the cubic symmetry of the grid geometry.

Finally at high T, Fig. 9*d*, one has approximately circular intensity rings, indicating that the floating charge lattice has melted into a charge liquid. The 4–fold asymmetry in these rings is due to the square geometry of the underlying grid. Similar results have been found by Hattel and Wheatley (1995), who work directly in the XY representation (1) with a cosine interaction.

In terms of the Josephson array, the pinned vortex lattice corresponds to a state in which the system is truly superconducting with a vanishing linear resistivity. In the floating lattice phase, the vortex lattice is free to diffuse as a whole, thus giving finite linear "flux flow" resistance in the presence of any applied d.c. current. The floating lattice phase is no longer truly superconducting. However the breaking of cubic symmetry due to the orientational coupling of the vortex lattice to the grid will lead to an anisotropic mobility for the vortex lattice. The result should be an angular dependent resistivity, and in the case that the applied current is not aligned with a symmetry direction, a non zero Hall voltage. Once the floating vortex lattice has melted, the cubic symmetry of the grid is restored, and one expects to see an isotropic resistivity with a vanishing Hall voltage. The vanishing of the Hall voltage therefore should serve as a clear experimental signal for the melting of the floating vortex lattice.

Simulations for other values of $f = 1/q$ yield the phase diagram shown in Fig. 11. $T_{\rm c}(f)$ denotes the transition between the pinned and floating lattices. As $f \to 0$, one sees that $T_{\rm c}(f) \sim f$ vanishes, in agreement with results by Nelson and Halperin (1979), Korshunov (1986), and Hattel and Wheatley (1994). $T_{\rm m}(f)$ denotes the melting temperature of the floating lattice. As $f \to 0$, $T_m \simeq 0.007$ approaches a finite constant. This value in agreement with estimates by Fisher (1980) for the melting of a vortex lattice in a continuous film. When the vortex density is too large, $f \gtrsim 1/30$, $T_{\rm c}$ and $T_{\rm m}$ merge, and

Fig. 11. Phase diagram for dilute densities $f = 1/q$

One can now ask about the nature of the depinning and melting transitions in these systems. The natural candidate for the melting transition is the theory of defect mediated melting in two dimensions, as developed by Kosterlitz and Thouless (1973), Nelson and Halperin (1979), and Young (1979) (KTNHY), and extended by Nelson and Halperin (1979) for a 2D solid on a periodic substrate. This KTNHY theory is essentially a modification of the $f = 0$ CG to *vector* charges, which are the Burger's vectors of lattice dislocations.

A two dimensional floating lattice is characterized by *quasi-long range*, or *algebraic*, translational order,

$$\langle \mathrm{e}^{\mathrm{i}\mathbf{G}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \rangle \sim |\mathbf{r}_i - \mathbf{r}_j|^{-\eta \ (T)} \ , \tag{12}$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are two charge positions, and $\mathbf{G}$ is a reciprocal lattice vector of the charge lattice. The translational correlation exponent $\eta_{\mathbf{G}}$ is related to the shear modulus μ of the charge lattice by $\eta_{\mathbf{G}} = T|\mathbf{G}|^2/4\pi(\mu + \gamma)$, where γ describes the coupling to the periodic substrate and we have used the fact that for logarithmically interacting charges the compression modulus $\lambda \to \infty$. A main prediction of the KTNHY theory is that upon heating, $\eta_{\mathbf{G}_1}$ ($\mathbf{G}_1$ is the smallest reciprocal lattice vector) takes a discontinuous jump to infinity at T_{m} from the universal value $\eta_{\mathbf{G}_1}(T_{\mathrm{m}}^-) = 1/3$. NH further show that the pinned commensurate to floating lattice transition is also described

[4] The numerical result that the floating lattice exits only for $f \lesssim f^* = 1/30$ compares with the estimate of Nelson and Halperin (1979) that $f^* \simeq 1/12$

Fig. 12. Translational correlation exponent η_{1} for the $f = 1/100$ CG on a triangular grid: (b) is an expanded scale of (a) focusing on the depinning transition

A second key prediction of the KTNHY melting theory for a lattice in a continuum is that above T_{m} there exists a distinct hexatic liquid phase with

[5] The assumption behind $\eta_{1}(T_{\mathrm{c}}^{+}) \sim 4f$, that couplings experience only small renormalizations, may not be valid here. If it were, one would expect that at low temperatures above T_{c}, one would have $\eta_{1} \sim T$. Although the curve $\eta_{1}(T)$ in Fig. 12 does look roughly linear between T_{c} and T_{m}, note that it does not extrapolate through the origin.

Fig. 13. Free energy histogram $F(E)$ at (a) melting transition and (b) depinning transition for various lattice sizes L, for the $f=1/49$ CG on a triangular grid

The issue of whether melting in two dimensions is described by KTNHY or is first order has remained hotly contested for more ordinary systems with short range interacting particles (for a review, see Strandburg 1988). Recent simulations (Bagchi et al. 1996) have suggested that only at very large system sizes does the hexatic liquid appear. In the present case, it may also be that the discreteness of the grid has preempted the KTNHY transition and made it first order. Hattel and Wheatley (1994) have argued that the depinning transition must become KTNHY–like at small enough densities f. However simulations of the one component CG in a continuum (Caillol et al. 1982, Choquard and Clerouin 1983), as well as other formulations of the vortex lattice melting problem (Tešanović and Xing 1991, Hu and MacDonald 1993, Kato and Nagaosa 1993, Šášik and Stroud 1994), all suggest that the melting transition is weakly first order.

2.4 Near Full Frustration: $f = 1/2 - 1/q$

For a density $f = 1/2 - 1/q$, q large, the ground state is almost everywhere like the $f = 1/2$ checkerboard pattern except with a superlattice of missing charges (*defects*). Consider here the specific case $f = 5/11$. The correct ground state for this case, shown in Fig. 2, was first found by Kolahchi and Straley (1991). The finite temperature behavior was first studied by by Franz and Teitel (1995). In Figs. 14*a-c* are shown intensity plots of the structure function $S(\mathbf{k})$ at three representative temperatures. At low T, Fig. 14*a*, one finds a periodic structure of sharp Bragg peaks. Note that the peaks in the corners arise from the $f = 1/2$–like background; these are brighter than the other peaks, which arise from the defect superlattice.

At an intermediate T, Fig. 14*b*, one continues to see sharp Bragg peaks in the corners, however the other peaks have been replaced by circular rings. The defect superlattice has melted into a defect liquid, but the $f = 1/2$–like background remains ordered. Finally, at high T, Fig. 14*c*, the peaks in the corners broaden, the $f = 1/2$–like background has melted, and one finds an isotropic liquid.

Looking more closely at Fig. 14*b* for the defect liquid, one sees that $S(\mathbf{k})$ is symmetric with respect to the Bragg planes that bisect the diagonals from the origin to the corners. This indicates that the defects, while freely diffusing, are still constrained to sit on only one sublattice of the original square grid of sites; equivalently, one never has two charges on two nearest neighbor sites. Since this sublattice has half the number of sites as the original grid, and since the defects interact with the same logarithmic interaction as do charges, the problem of $f = 1/2 - 1/q$ at low temperatures becomes equivalent to that considered in the previous section: the $f = 1/2$–like background remains ordered and can be ignored; the dilute density of mobile defects behaves in the same way as a dilute density of charges with $f' = 2/n$. This leads to the expectation that, for q sufficiently large, one will find an additional phase not observed for $f = 5/11$. Upon heating, one will first have a transition T_c from

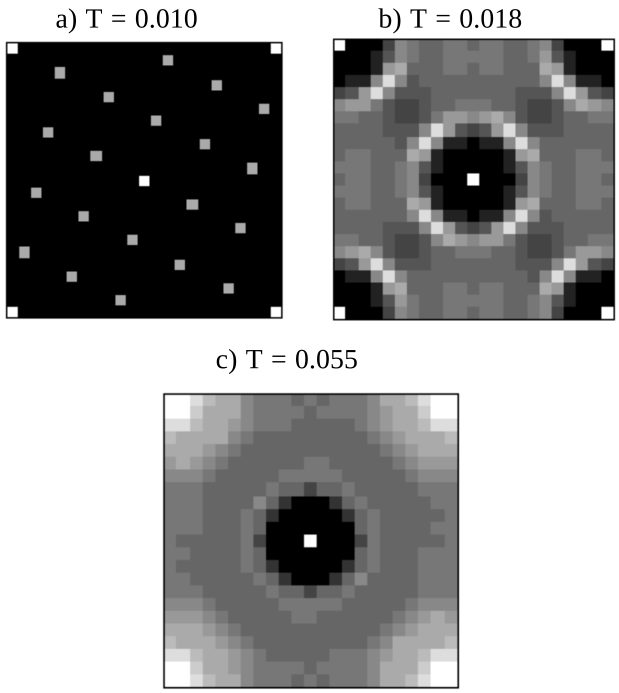

Fig. 14. (a) Structure function $S(\mathbf{k})$ for the case $f = 5/11$ at (a) $T = 0.010$, (b) $T = 0.018$, and (c)$T = 0.055$

a pinned to a floating defect superlattice, followed by a melting transition T_{m} to a defect liquid, followed finally by the melting $T_{\mathrm{m}'}$ of the $f = 1/2$–like background into an isotropic liquid. The phase boundaries for T_{c} and T_{m} may be inferred from Fig. 11. The boundary for $T_{\mathrm{m}'}$ remains in general unknown. The depinning transition T_{c} will mark the vanishing of ϵ^{-1} and hence the transition between the superconducting and the normal states in the Josephson array.

2.5 General *f*

For general rational $f = p/q$, Teitel and Jayaprakash (1983b) (TJ) presented arguments that the loss of superconductivity in a Josephson array would occur at a temperature $T_{\mathrm{c}}(f) \sim 1/q$. This argument was based on considering the effect of applying a uniform twist gradient δ to the phases θ_i of the model (1), and studying the periodicity of resulting supercurrents as δ is varied. However, for large systems $L \to \infty$, the maximum twist gradient

Fig. 15. Schematic of a ground state for $f = 1/3 - 1/q$. ($\bullet$) denotes a charge; a shaded box denotes a charge missing from the 1/3–like background. At low T, moves of type "1" are more likely than moves of type "2".

[6] Fig. 15 is a schematic only. Straley and Barnett (1993) note that, for $f = 1/3 - 1/q$, one expects the defect superlattice to be the correct ground state only at *larger* values of q than depicted in Fig. 15.

One can easily imagine similar scenarios near other simple rational fractions. If $f = p/q$ is near some simpler $f_0 = p_0/q_0$, then the ground state of f will look almost everywhere like that of f_0, with a superlattice of defect regions. When $f - f_0$ is sufficiently small, these defects should unpin at a $T_c(f) \ll T_c(f_0)$, thus leading to a very discontinuous phase boundary $T_c(f)$. However the detailed dependence of T_c on f remains in question. In particular, for the dilute density $f = 1/q$ it has been shown that $T_c \sim 1/q$. It would be interesting to see if the depinning transition for a dilute density $f = p/q$, $p \ll q$, is proportional to the charge density p/q, or if there are additional commensurability effects that cause $T_c(p/q) \sim 1/q$ as in the TJ conjecture.

A related question is the behavior at irrational values of f. Approximating an irrational f by a very high order rational p/q, $q \gg 1$, TJ argued that $T_c(f) \to 0$. The specific case of $f = (3 - \sqrt{5})/2$ has been studied numerically by Halsey (1985,1988) who argued in favor of a finite T_c. Halsey's evidence consists of the observation that (*i*) the Josephson array appears to have a finite zero temperature critical current, and (*ii*) that an Edwards-Anderson-like order parameter appears to be finite at low temperature. However it has been shown (Lobb et al. 1983, Straley 1988, Rzchowski et al. 1990, Vallat and Beck 1992) that a finite $T = 0$ critical current persists even for the case of a single vortex, i.e. $f = 1/L^2$. This is just a single body effect of the vortex moving in an effective two dimensional periodic pinning potential created by the grid of the array. At any finite T however, thermal activation over the energy barriers of this single body pinning potential will lead to finite linear resistivity, in agreement with the observation that for $f = 1/q$, $T_c \sim 1/q \to 0$ as $q = L^2 \to \infty$. Thus (in contrast to TJ's assumption) a finite $T = 0$ critical current does not imply a finite T_c. A finite Edwards-Anderson order parameter at low T would be a more convincing demonstration of a finite T_c. However Halsey leaves open the question of whether his results reflect a true equilibrium transition, or rather a freezing out of equilibrium due to a finite cooling rate in the presence of large energy barriers between many metastable nearly degenerate states. Thus behavior at irrational f remains an open question.

3 Arrays with Randomness

The introduction of randomness into the Josephson array, while posing many new challenges for theory, is also a topic of great practical importance. As vortex diffusion is a source of "flux flow" resistivity in superconductors, the introduction of random vortex pinning impurities has been viewed as a way of increasing critical currents and enhancing superconductivity. On the other hand, introducing randomness into statistical models often has the effect of reducing the transition temperature, or even of driving $T_c \to 0$. In this section I will review several simple models in which the effects of randomness have been included in the Josephson array.

3.1 Random Point Pinning

A topic that has received renewed interest in connection with behavior in high T_c superconductors, has been the effect of random point pinning sites on the structure of a vortex lattice. This question can be easily addressed within the CG model for a Josephson array, by adding a random potential to the Hamiltonian (3),

$$\mathcal{H}[n_i] = \tfrac{1}{2}\sum_{i,j}(n_i - f)G(\mathbf{r}_i - \mathbf{r}_j)(n_j - f) + \sum_i V_i n_i \ . \tag{14}$$

Here V_i is a random pinning potential on each site of the grid, with averages,

$$\overline{V_i} = 0, \qquad \overline{V_i V_j} = w^2 \delta_{ij} \ , \tag{15}$$

where the overbar denotes an average over different realizations of the random potential V_i. Note that V_i can be viewed as arising from random magnetic fluxes δf_j, $V_i = \sum_j G(\mathbf{r}_i - \mathbf{r}_j)\delta f_j$. The δ_{ij} correlations of (15) then imply long range correlations between these random fluxes, $\overline{\delta f_i \delta f_j} = \sum_m G^{-1}(\mathbf{r}_i - \mathbf{r}_m)G^{-1}(\mathbf{r}_m - \mathbf{r}_j)$, were $G^{-1}(\mathbf{r})$ is the inverse of the Green's function, proportional to the two dimensional lattice Laplacian, $G^{-1}(\mathbf{r}) = \frac{1}{2\pi}\sum_\mu [\delta_{\mathbf{r},\hat{\mu}} - \delta_{\mathbf{r},0}]$, where $\mu = \pm x, \pm y$. In contrast, localized randomness in a bond parameter of (1), for example A_{ij} or a bond coupling J_{ij}, would in general lead to a longer range correlation in the pinning potential V_i (Cohn et al. 1991). Equation (14) is thus an idealized model of random pinning, rather than a true model of the effects of randomness in a physical Josephson array. For a more realistic model treating the effects of bond dilution on a Josephson array, see Li and Teitel (1991).

The effects a of random point pinning potential as in (14) have been treated in a classic paper by Larkin and Ovchinnikov (1979). They argue that any amount of random pinning will lead to exponentially decaying translational correlations on long enough length scales (see also Chudnovsky 1991). The length scale L_p on which the random pins disorder the vortex lattice is estimated as follows. In a domain of size L, there will be $M = (L/a_\mathrm{v})^d$ vortices, where a_v is the average distance between vortices. If the vortices in the domain are still ordered, then the total pinning energy felt by these M vortices will be the sum of M uncorrelated values of V_i. The root mean square average of this pinning energy is,

$$U_\mathrm{pin} = \sqrt{M w^2} = \left(\frac{L}{a_\mathrm{v}}\right)^{d/2} w \ , \tag{16}$$

where d is the spatial dimension. The competing elastic energy to distort the vortices in the domain by one lattice constant a_v over the length L is approximately,

$$U_\mathrm{el} = \tfrac{1}{2}\mu \left(\frac{a_\mathrm{v}}{L}\right)^2 L^d \ , \tag{17}$$

where μ is the shear modulus of the vortex lattice. The domain will remain ordered provided $U_{\rm pin} < U_{\rm el}$. The criteria $U_{\rm pin} = U_{\rm el}$ thus determines the length $L_{\rm p}$ beyond which elastic distortions destroy the long range order of the vortex lattice,

$$\frac{L_{\rm p}}{a_{\rm v}} = \left(\frac{\mu a_{\rm v}^d}{2w}\right)^{2/(4-d)} = \frac{\mu a_{\rm v}^2}{2w} \quad \text{for } d=2 \ . \tag{18}$$

Using the KTNHY criteria for the vortex lattice melting transition, $\eta_{\mathbf{G}_1} = 1/3$ (which by Fig. 12 appears to be well obeyed, even if the transition is weakly first order), one gets $\mu a_{\rm v}^2 = 4\pi T_{\rm m}$, yielding $L_{\rm p}/a_{\rm v} = 2\pi T_m/w$.

More recently, Giamarchi and Le Doussal (1994), and Korshunov (1993), have challenged the Larkin-Ovchinnikov result. Their variational renormalization group analysis finds that at low temperatures disorder does not lead to exponential decay, but rather to algebraically decaying correlations, as in the case of the pure (non-random) vortex lattice in two dimensions. The translational correlation exponent $\eta_{\mathbf{G}_1}$ of this algebraic decay, instead of vanishing linearly in T as in the pure case, now saturates at low T to a finite constant, independent of the strength of the disorder w.

Both the Larkin-Ovchinnikov and variational RG analyses treat only elastic distortions, ignoring the possibility of disorder induced free dislocations. Such dislocations are expected to appear, and lead to exponentially decaying translational correlations, on some length scale $L_{\rm d}$. One can then ask how the length $L_{\rm d}$ compares to the length $L_{\rm p}$, and in the case that $L_{\rm p} \ll L_{\rm d}$, how do the correlations decay in the intermediate region $L_{\rm p} < L < L_{\rm d}$.

In order to investigate this question, Franz and Teitel (1996) have carried out Monte Carlo simulations of the Hamiltonian (14) for the $f = 1/100$ CG on a triangular grid of length $L = 100$. In Fig. 16 below are shown the results for $\eta_{\mathbf{G}_1}$ vs. T, for several values of pinning strength w, as determined by the same procedure as used in Fig. 12. In agreement with the variational RG calculations, for temperatures smaller than the pure case $T_{\rm m}$, one has a good fit to an algebraic decay. Upon cooling, $\eta_{\mathbf{G}_1}(T)$ tracks the pure case value, until saturating to a constant at low temperature. However, in opposition to the variational RG results, this low temperature value continues to increase with increasing w. For the largest value of w shown in Fig. 16, $\eta_{\mathbf{G}_1}$ stays constant at the critical value of $1/3$ for all $T < T_{\rm m}$.

In Fig. 17 are shown results for $\eta_{\mathbf{G}_1}$ vs. w, for fixed $T = 0.004$, midway between the pure case melting and depinning transitions. Comparing increasing with decreasing w for $L = 100$, one sees a strong hysteresis as $\eta_{\mathbf{G}_1}$ crosses the critical KTNHY value of $1/3$. $\eta_{\mathbf{G}_1} = 1/3$ signals the appearance of disorder induced free dislocations. For comparison, the same results are shown for the smaller system length $L = 50$. While again one finds a good fit to algebraic decay in the dislocation free region $\eta_{\mathbf{G}_1} < 1/3$, the exponent $\eta_{\mathbf{G}_1}$ is now found to be significantly smaller than the value found for $L = 100$. If correlations truly decayed algebraically, one would expect the results from

Fig. 17. Translational correlation exponent η_1 vs. disorder strength w for $f = 1/100$ and $L = 100, 50$

Finally, Franz and Teitel find that for all temperatures $T_c^{\rm pure} < T < T_m^{\rm pure}$, $\epsilon^{-1} = 0$. Thus the random pinning potential, in addition to disordering the vortex lattice, fails to pin vortices to the substrate.

3.2 Random Gauge Models

Finally, I will mention some other random models that have received attention recently. These are all models in which the variables A_{ij} of the Hamiltonian (1) are taken as random variables, hence the term *random gauge* model.

XY Spin Glass: If one takes the A_{ij} to be independent random variables with equally likely values 0 and π, this corresponds to the ordinary XY model with equally likely ferromagnetic and antiferromagnetic bond couplings. This is the two dimensional XY *spin glass* for which it has been established that $T_c = 0$. Nevertheless, an interesting complexity remains. Just as in the $f = 1/2$ case there existed the possibility of separate transitions with respect to the continuous and discrete symmetries, here it appears that the two correlation lengths, that describe the phase and vortex (chiral) orderings as $T \to 0$, diverge with distinctly different exponents (Kawamura and Tanemura 1987, Ray and Moore 1992, Bokil and Young 1996).

Gauge Glass: If one takes the A_{ij} to be independent random variables uniformly distributed on the interval $(-\pi, \pi]$, this corresponds to independently correlated random magnetic fluxes through each unit cell of the array, and is referred to as the *gauge glass*. Here again it appears that $T_c = 0$ (Fisher et al. 1991, Cieplak et al. 1992, Gingras 1992).

Gaussian Phase Shifts: If one takes the A_{ij} to be independent Gaussian distributed random variables, with average zero and standard deviation σ, this is known as the *random Gaussian phase shift* model, or also the *positionally disordered* Josephson array. The latter name refers to one way such a model can be physically realized. If the position of either a node or a bond of the Josephson array (see Fig. 1) is randomly displaced, then in the presence of a uniform applied magnetic field, there will be a randomly shifted value for A_{ij} on the effected bonds. This type of disorder corresponds to the case of random quenched dipole fluxes, $\delta f_i = +\epsilon$ and $\delta f_j = -\epsilon$, on nearest neighbor sites i and j. To see this, note that changing A_{ij} will increase the magnetic flux through the unit cell on one side of the bond $\langle ij \rangle$, while decreasing the flux by the same amount through the unit cell on the opposite side of the bond. For a given positionally disordered array, one can increase the strength of the randomness by increasing the uniform applied magnetic field by an amount equal to an integer number of flux quanta per unit cell. This has no effect on the average fluxes f_i (which can always be shifted back to the interval $(-\frac{1}{2}, \frac{1}{2}]$ leaving all physical quantities invariant), but increases the strength ϵ of the quenched random fluxes, δf_i.

For very large σ, this model reduces to the gauge glass, and $T_c = 0$. For $\sigma < \sigma_c$ however, a finite temperature transition has been predicted. This model was first analyzed by Rubinstein et al. (1983), and applied to the Josephson array problem by Granato and Kosterlitz (1986). For the case where the average f_i is integer (or equivalently zero), they find that as T is decreased for $\sigma < \sigma_c$, there is first a KT transition to an ordered state, followed at lower temperature by a reentrant transition back to the disordered state. Numerical simulations by Forrester et al. (1988, 1990) and by Chakrabarti and Dasgupta (1988), as well as experiments (Forrester et al. 1988, Benz et al. 1988), failed to find any evidence for a reentrant phase. More recently, Ozeki and Nishimori (1993) have presented results which claim to rigorously rule out such a reentrant phase, but leave open the question of whether there remains a finite ordering transition T_c, or if $T_c \to 0$. Most recently, Nattermann et al. (1995), Scheidl (1996), and Tang (1996), all report arguments that a finite KT type transition should exist. This transition in the $\sigma - T$ plane should lie near the KT transition line as found by Rubinstein et al. The reentrant transition of Rubinstein et al. disappears, but is replaced by a cross-over region below which behavior becomes glassy. Numerical simulations that confirm these new results remain to be done.

The case where random Gaussian phase shifts are superimposed upon a fractional average f_i (Choi et al. 1987), remains a largely unexplored problem.

4 Conclusions

To conclude, I have attempted to review some of the rich equilibrium critical phenomena exhibited by classical two dimensional arrays of Josephson junctions. Many interesting old and new questions remain to be resolved. In particular, it would be interesting to have a better understanding of how frustration induced by quenched randomness, as in Sect. 3, differs from frustration induced by geometrical effects, such as in the case of uniform irrational f, or the $f = 1/2$ CG on a triangular grid. The former are more akin to spin glass problems, while the latter appear more akin to the structural glass problem.

I have omitted mention of many other interesting areas of ongoing research on Josephson junction arrays. In the area of equilibrium behavior of classical arrays these include such topics as fractal, incommensurate, anisotropic, and three dimensional geometries. In the area of dynamics these include, understanding the critical behavior of I-V characteristics, giant Shapiro steps, mode locking, chaos, the contrasting behavior of underdamped as opposed to overdamped junctions, and plastic flow of vortex lattices. Quantum effects become important when charging energy must be considered. A good review of recent work on many of these topics may be found in Giovannella and Tinkham (1995).

Clearly the study of Josephson junction arrays will continue to yield much new interesting physics well into the future.

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