Maximum Entropy and the Stress Distribution in Soft Disk Packings Above Jamming

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We show that the maximum entropy hypothesis can successfully explain the distribution of stresses on compact clusters of particles within disordered mechanically stable packings of soft, isotropically stressed, frictionless disks above the jamming transition. We show that, in our two dimensional case, it becomes necessary to consider not only the stress but also the Maxwell-Cremona force-tile area, as a constraining variable that determines the stress distribution. The importance of the force-tile area was suggested by earlier computations on an idealized force-network ensemble.

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As the density of deformable granular particles increases above a critical packing fraction, \( \phi_c \), the system undergoes a jamming transition from a liquid-like to a solid-like state \([1, 2]\). For large particles thermal fluctuations are irrelevant, and in the absence of mechanical agitation, the dense system relaxes into a mechanically stable rigid but disordered configuration. Given a set of macroscopic constrains there are in general a large number of such configurations accessible. Edwards and co-workers \([3]\) first proposed that it was possible to give a statistical description of such quenched configurations using the methods of statistical mechanics. Henkes and co-workers \([4, 5]\) extended these ideas to the ensemble, describing the distribution of stresses within the jammed packing. Tighe and co-workers \([6–8]\) using an idealized force-network ensemble on a periodic triangular lattice argued that, in two dimensions (2D), the Maxwell-Cremona force-tile area acts as an additional constraining variable that effects the stress distribution. Recent experiments \([9–11]\) have sought to test such statistical models.

Here we consider the stress distribution on compact clusters of particles, within isotropically stressed static packings of soft-core disks in 2D, showing that it is well described by the maximum entropy hypothesis. However, in contrast to earlier work \([1]\), we show that it is necessary to take both stress and force-tile area as constraining variables in order to characterize the observed distribution.

Our system is a bidisperse mixture of equal numbers of big and small circular, frictionless, disks with diameter ratio 1.4 \([2]\). Particles \(i\) and \(j\) interact only when they are in contact, and then they repel with a harmonic interaction potential, \(V(r_{ij}) = \frac{1}{2}k_c (1 - r_{ij}/d_{ij})^2\), where \(r_{ij}\) is the center-to-center distance between the particles, and \(d_{ij}\) is the sum of their radii. We consider here packings with a fixed total number \(N\) of disks, that have an isotropic total stress tensor \(\Sigma^{(N)} = N \delta_{\alpha\beta}\), where \(N = pV\) is given by the pressure \(p\) and the total system volume \(V\) (in 2D we will use “volume” as a synonym for area). Here \(\alpha, \beta\) denote the spatial coordinate directions \(x, y\). We will measure energy in units such that \(k_c = 1\), and length in units such that the small disk diameter is unity. Lees-Edwards boundary conditions \([12]\) are used to periodically replicate our simulation box. To create our packings we start with random initial particle positions and then quench to a local energy minimum. During minimization, the box geometry is varied \([13]\) to ensure that the resulting mechanically stable state has a strictly isotropic stress tensor with the desired \(\Gamma_N\). Further details of our procedure may be found in Ref. \([14]\). Unless stated otherwise, our results are for a system with \(N = 8192\) disks, averaged over 10,000 independently generated isotropic configurations.

Since our simulations fix both \(N\) and \(\Gamma_N\), it is convenient to parameterize our results by the intensive, pressure-like, variable, \(\tilde{p} \equiv \Gamma_N/N = pV/N\), the total stress per particle. Since our method varies the system volume \(L_x L_y\), so as to achieve the desired total stress \(\Gamma_N\), the packing fraction \(\phi\) for a fixed \(\Gamma_N\) varies slightly from configuration to configuration. In Fig. 1, we plot the resulting average \(\langle \tilde{p} \rangle\) as a function of \(\tilde{p}\) for the range of \(\tilde{p}\) considered in this work. Error bars represent the width of the distribution of \(\phi\); the relative width is roughly 0.03 – 0.04%. The \(\tilde{p}\) values we consider here are all close above the jamming transition, which for our rapid quench protocol is \(\phi_c \approx 0.842\) for an infinite system \([15]\).

To define our particle clusters, we pick a point at random and draw a circle of radius \(R\); all particles whose centers lie within this circle are considered part of the cluster \([16]\). In the lower inset to Fig. 1 we plot the average number of particles \(N_R\) expected in such a cluster of radius \(R\).

We now compute the stress tensor for this cluster \(C_R\),

\[
\Sigma^{(R)}_{\alpha\beta} = \sum_{i \in C_R} \sum_j s_{ij} F_{ij\alpha} \delta_{\alpha\beta}, \quad F_{ij} = -\partial V/\partial r_{ij},
\]

The first sum is over all particles \(i\) in the cluster \(C_R\). The second, primed, sum is over all particles \(j\) in contact with \(i\), where \(s_{ij}\) is the displacement from the center of particle \(i\) to its point of contact with \(j\), and \(F_{ij}\) is the force on \(j\) due to contact with \(i\) \([4]\).

Although the total system stress is isotropic, the stress on any particular cluster \(\Sigma^{(R)}_{\alpha\beta}\) in general is not. However
the stress averaged over many independent clusters will be isotropic. If we define $\Gamma_R \equiv \frac{1}{2} \text{Tr}|\Sigma(R)|$, then we have

$$\langle \Sigma_{\alpha\beta}(R) \rangle = \langle \Gamma_R \rangle \delta_{\alpha\beta}. \quad (2)$$

For the particular case $R = 5.4$ we plot in Fig. 1b the numerically computed probability histograms $P(\Gamma_R | \bar{\rho})$ that the cluster has a particular value $\Gamma_R$ at a given value of the total system stress per particle $\bar{\rho}$.

Noting that $\Sigma_{\alpha\beta}$ is a conserved quantity (i.e. its global value is fixed and it is additive over disjoint subsystems) Henkes et al. [4, 5] proposed the stress ensemble in analogy to the canonical ensemble of statistical mechanics; $\Gamma_R$ for isotropic systems plays the role of energy, and the distribution of $\Gamma_R$ was proposed to be,

$$P(\Gamma_R | \bar{\rho}) = \Omega_R(\Gamma_R) \frac{e^{-\alpha \bar{\rho} \Gamma_R}}{Z_R(\bar{\rho})}. \quad (3)$$

The angoricity [5, 17] $1/\alpha$ is a temperature-like variable that is set by the total system stress $\bar{\rho}$; the number of states $\Omega_R$ is presumed independent of $\bar{\rho}$, and $Z_R$ is a normalizing constant. The same distribution results from a maximum entropy hypothesis [18], in which all clusters with a given $\Gamma_R$ are presumed equally likely, and the average is constrained to the known value $\langle \Gamma_R \rangle$. Since $\Gamma$ is conserved and additive, its average is constrained by,

$$\langle \Gamma_R \rangle = \Gamma_N \left( \frac{\pi R^2}{V} \right), \text{ or } \langle p_R \rangle \equiv \langle \Gamma_R \rangle \frac{\pi R^2}{V} = \frac{\Gamma_N}{V} = p, \quad (4)$$

a result that we have confirmed numerically [14].

Denoting quantities at a given $\tilde{p}_1$ or $\tilde{p}_2$ by the subscript 1 or 2, the log ratio of histograms at two neighboring values of $\tilde{p}_1 < \tilde{p}_2$ is given by [19],

$$\ln \left[ \frac{P_1}{P_2} \right] - \ln \left[ \frac{Z_{R,2}}{Z_{R,1}} \right] = (\alpha_2 - \alpha_1) \Gamma_R \equiv \Delta \alpha \Gamma_R. \quad (5)$$

Since the right hand side of Eq. (5) scales proportional to the cluster area $\pi R^2$, so must the left hand side. We therefore define an intensive scaled log ratio,

$$R_s = \frac{1}{\pi R^2} \left( \ln \left[ \frac{P_1}{P_2} \right] - \ln \left[ \frac{Z_{R,2}}{Z_{R,1}} \right] \right) = \Delta \alpha p_R. \quad (6)$$

where $p_R \equiv \Gamma_R / (\pi R^2)$.

Since we expect $\alpha(p)$ to be an intensive parameter, independent of cluster size for sufficiently large $R$ [20], plotting $R_s$ vs $p_R$ for a given pair of $\tilde{p}_1$ and $\tilde{p}_2$, vs all cluster sizes $R$ should collapse to a common curve. Taking $c_R \equiv \ln[Z_{R,2}/Z_{R,1}]$ as an adjustable parameter, we find that we can indeed collapse data for different $R$ in this way. In Fig. 2 we show results for several different pairs of $\tilde{p}_1$ and $\tilde{p}_2 = \tilde{p}_1 + 9.8 \times 10^{-5}$, collapsing data for cluster sizes $R = 2.8 - 7.8$, containing roughly $N_R = 18 - 139$ particles. Our largest $R$ is small enough that effects due to our finite total system size are negligible.

While the collapses are excellent, the data show a clear curvature, not the linearity predicted by Eq. (6). If we for the moment ignore this curvature, we can approximate $R_s$ by the tangent line at the value of $p_R$ where the two distributions $P_1$ and $P_2$ have their largest overlap (dashed lines in Fig. 2b). Defining $\Delta \alpha$ as the slope of this tangent line, $\Delta \alpha / \bar{\rho}$ vs $\bar{\rho}$ is the solid line is the best fit to the form $a / \bar{\rho}^2$, and gives $a \approx 2.3$. Taking $-\Delta \alpha / \bar{\rho}$ as an approximation to the derivative, we integrate to get $\bar{\alpha} = 2.3 / \bar{\rho} = 2.3(N/\Gamma_N)$, in reasonable agreement with Henkes et al. [4] who, looking at a somewhat different distribution ratio, claimed agreement with the linear form of Eq. (6) and found $\bar{\alpha} = 2(N/\Gamma_N)$. Note, however, if we fit our $-\Delta \alpha / \bar{\rho}$ to a general power law, $a / \bar{\rho}^q$, our data is better fit by the value $q \approx 1.9$ [21].
The tangent lines, however, are a poor description of our data, which instead are well fit by parabolas (solid lines in Fig. 2). We thus have,

$$R_s = \Delta \alpha p_R + \Delta \lambda \bar{p}_R^2$$  \hspace{1cm} (7)

with $\Delta \alpha$ and $\Delta \lambda$ independent of the cluster radius $R$. In Fig. 3 we plot $\Delta \alpha / \Delta \bar{p}$ and $-\Delta \lambda / \Delta \bar{p}$ vs $\bar{p}$, and find that they scale like $\sim \bar{p}^{-q}$ with $q \approx 1.9$ and 2.9 respectively \[21\].

![FIG. 3. (color online) Scatter plot of values of stress $\Gamma_R$ and force-tile area $A_R$, for clusters of radius $R = 5.4$, for different values of total system stress per particle $\bar{p}$ ranging from 0.00088 to 0.00225. The smaller the value of $\bar{p}$, the more compact is the distribution.](image)

To explain the curvature in our data of Fig. 2 we turn to the work of Tighe et al. \[22\], who, using an idealized ordered triangular lattice to model a force network, realized that in 2D granular packings there is another constraining variable, in addition to the stress $\Gamma$, that can effect the stress distribution; this is the Maxell-Cremona force-tile area $A$. For disks in a mechanically stable packing, the force-tile for particle $i$ is obtained by rotating all contact forces $90^\circ$ and lying them tip to tail. Force balance then requires these to form a closed loop \[22\]. The area $A_i$ of this loop is the force-tile area. It can be shown that such force-tiles tile space with no gaps or overlaps \[7\]. The force-tile area of a cluster of particles $C$ is then just the sum over force-tile areas for each member particle, $A_C = \sum_{i \in C} A_i$.

For a system with periodic boundary conditions, the total system force-tile area $A_N$ is determined precisely by the total system stress $\Gamma_N$, with $A_N = \Gamma_N^2 / V$ \[7\]. However for clusters of radius $R$, since the boundary is not fixed, $A_R$ may take a distribution of values for each given value of $\Gamma_R$. We illustrate this in Fig. 3 where we show a scatter plot of the values of $A_R$ and $\Gamma_R$ found in individual clusters, for the particular case $R = 5.4$ at several different values of the total system stress per particle $\bar{p}$.

Since the force-tile area is conserved (i.e. the total system value $A_N$ is fixed and $A$ is additive over disjoint subsystems) the average on clusters of radius $R$ is constrained by \[13\],

$$\langle A_R \rangle = A_N \left( \frac{\pi R^2}{V} \right).$$  \hspace{1cm} (8)

Combining the above with Eq. (4), and using the fixed relation between $A_N$ and $\Gamma_N$, then yields the relation between average cluster force-tile area and average cluster stress, $\langle A_R \rangle = (\Gamma_R)^2 / \pi R^2$. Thus a maximum entropy formulation should consider the joint distribution of both $\Gamma_R$ and $A_R$, treating both as constrained variables whose averages are known. Assuming all configurations with a given pair of $(\Gamma_R, A_R)$ are equally likely, one gets,

$$\mathcal{P}(\Gamma_R, A_R | \bar{p}) = \Omega_R(\Gamma_R, A_R) e^{-(\bar{p})\Gamma_R - \lambda(\bar{p})A_R} / Z_R(\bar{p}).$$  \hspace{1cm} (9)

Considering distributions at two neighboring values of the total system stress per particle, $\bar{p}_1$ and $\bar{p}_2$, we can again construct the scaled log histogram ratio,

$$R_s = \frac{1}{\pi R^2} \left( \ln \left[ \frac{P_1}{P_2} \right] - \ln \left[ \frac{Z_{R,2}}{Z_{R,1}} \right] \right) = \Delta \alpha p_R + \Delta \lambda a_R$$  \hspace{1cm} (10)

where $\Delta \alpha = \alpha_2 - \alpha_1$, $\Delta \lambda = \lambda_2 - \lambda_1$, and $a_R = A_R / \pi R^2$. Thus plotting $R_s$ vs the intensive quantities $p_R$ and $a_R$, data for different cluster sizes $R$ should all collapse to a single plane for a given pair $\bar{p}_1, \bar{p}_2$. The slopes of the plane in directions $p_R$ and $a_R$ determine the values of $\Delta \alpha$ and $\Delta \lambda$.

Again, choosing $c_R \equiv \ln[Z_{R,2}/Z_{R,1}]$ as adjustable parameters, we find that our data for $R_s$ do indeed collapse to a single plane for all $R$ \[23\]. In Fig. 4 we show $R_s$ vs $p_R$ and $a_R$ for several different cluster radii $R$, at the particular value $\bar{p}_1 = 0.00078$. Fig. 4a shows a side view looking down upon this plane; the data cluster into more compact regions as $R$ increases. Fig. 4b shows a view looking edge on at the plane, thus confirming that the surface defined by our data is indeed a flat common plane for all $R$. In Fig. 4c we show the chi squared per degree of freedom, $\chi^2 / \nu$, of the fit of our data to Eq. (10), as a function of $\bar{p}$. We show $\chi^2 / \nu$ for the entire fit using all cluster sizes $R$, as well as the $\chi^2 / \nu$ for subsets of data corresponding to a few different values of $R$. We see that the fit, with $\chi^2 / \nu \approx 1$, is quite satisfactory for all cases. In Fig. 5 we plot the resulting $\Delta \alpha / \Delta \bar{p}$ and $-\Delta \lambda / \Delta \bar{p}$ vs $\bar{p} = (\bar{p}_1 + \bar{p}_2) / 2$. We compare results from $N = 8192$ with results from a smaller system $N = 1024$ to show that the finite size of our total system has little effect on our extracted $\Delta \alpha$ and $\Delta \lambda$. We find $\Delta \alpha / \Delta \bar{p}$ and $\Delta \lambda / \Delta \bar{p}$ scale as $\bar{p}^{-q}$ with $q \approx 1.8$ and 3.0 respectively \[21\].

Finally we relate our results from the analysis of $\mathcal{P}(\Gamma_R, A_R)$ to our earlier results in Fig. 2. We can obtain the distributions of Fig. 1 by integrating over the joint distribution of Eq. (4), $\mathcal{P}(\Gamma_R | \bar{p}) = \int dA_R \mathcal{P}(\Gamma_R, A_R | \bar{p})$. If the number of states $\Omega_R(\Gamma_R, A_R)$ is strongly peaked
confirming that all data lies on a common flat plane. (c) Chi-

distribution is described by a simple Gaussian, rather

In all cases, \( \chi^2/\nu \approx 1 \) indicates a good fit.

about \( A_R = \Gamma^2/\pi R^2 \), then we can approximate \( \mathcal{P}(\Gamma_R \mid \tilde{p}) \approx \mathcal{P}(\Gamma_R, \Gamma^2/\pi R^2 \mid \tilde{p}) \). In that approximation one finds the \( \mathcal{R}_s \) obtained from \( \mathcal{P}(\Gamma_R \mid \tilde{p}) \) to have the form of Eq. (7), with \( \alpha \) and \( \lambda \) the same as defined by Eq. (9). In Fig. 6 we compare \( \Delta \alpha \) and \( \Delta \lambda \) as obtained directly from the joint distribution \( \mathcal{P}(\Gamma_R, A_R \mid \tilde{p}) \) and the fit to Eq. (10), with the values obtained from \( \mathcal{P}(\Gamma_R \mid \tilde{p}) \) and the fit to Eq. (7). The agreement for \( \Delta \lambda \) is very good; \( \Delta \alpha \), however, shows a systematic shift between the two methods, which we attribute to the finite width of \( \Omega_R(\Gamma_R, A_R) \) in variable \( A_R \) about its maximum for fixed \( \Gamma_R \).

We have also considered the possibility that our stress distribution is described by a simple Gaussian, rather than the Boltzmann-like distribution obtained from the maximum entropy hypothesis. However we find the Boltzmann-like distribution gives a measurably better fit to our data. See our Supplemental Material [21].

To conclude, we find that the distribution of stress in finite clusters of frictionless granular particles in an isotropic, mechanically stable, configuration above jamming is well described by the maximum entropy hypothesis, provided one identifies all relevant constraining variables, in this case \( \Gamma_R \) and \( A_R \). One may ask if there might not be other constraining variables that effect the distribution of \( \Gamma_R \) or \( A_R \). In particular, the total number of particles in the cluster \( N_R \), the number of small particles \( N_{s,R} \), and the Voronoi volume \( V_R \) of the cluster are also all conserved quantities with known average values: \( \langle N_R \rangle = N(\pi R^2/V) \), \( \langle N_{s,R} \rangle = (N/2)(\pi R^2/V) \), \( \langle V_R \rangle = \pi R^2 \). However we have found [13] that the correlations between these variables and \( \Gamma_R \) are considerably smaller than that between \( \Gamma_R \) and \( A_R \), and that these correlations decrease as \( R \) increases. This observation, as well as the goodness of our fits in Fig. 4 leads us to conclude that an adequate description is given by \( \Gamma_R \) and \( A_R \) alone.

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**SUPPLEMENTAL MATERIAL**

**Gaussian Assumption**

In these notes we consider an alternative possibility, that the distribution of stress on clusters is given by a simple Gaussian distribution. We show that this assumption yields a poorer explanation of our data than does the Boltzmann-like distribution of Eq. (9) of the main article.

First we consider the distribution of stress \( \Gamma_R \) alone. We will assume a Gaussian probability distribution,

\[
P(\Gamma_R) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{(\delta\Gamma_R - \langle \Gamma_R \rangle)^2}{\sigma^2}}
\]

where \( \delta \Gamma_R \equiv \Gamma_R - \langle \Gamma_R \rangle \), and \( \sigma^2 \) is the variance of \( \Gamma_R \). Both \( \langle \Gamma_R \rangle \) and \( \sigma \) are functions of the total system stress per particle \( \tilde{p} \). If we then consider the log ratio of distributions at two neighboring values \( \tilde{p}_1 \) and \( \tilde{p}_2 \) we get,

\[
\ln \left( \frac{\mathcal{P}_1}{\mathcal{P}_2} \right) = c + a \Gamma_R + b \Gamma_R^2,
\]
where

\[ c \equiv \ln \left( \frac{\sigma_2}{\sigma_1} \right) + \frac{(\Gamma_R)_2^2}{2\sigma_2^2} - \frac{(\Gamma_R)_1^2}{2\sigma_1^2} \]  
\[ a \equiv \frac{(\Gamma_R)_1}{\sigma_1^2} - \frac{(\Gamma_R)_2}{\sigma_2^2} \]  
\[ b \equiv \frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2} \]

where the subscript 1, 2 refers to values at \( \tilde{p}_1, \tilde{p}_2 \). We thus see that the assumption of a Gaussian distribution of stresses provides an easy (possible) explanation for the curvature seen in the log ratios shown in Fig. 2 of the main article.

Since we can easily compute averages and variances of \( \Gamma_R \), the result of Eq. (S2) involves no adjustable parameters, and we can directly see how well it agrees with our numerically computed values for the log ratio. In Figs. S1a and b we plot our data together with the prediction of Eq. (S2) (solid lines) for two different cluster radii, \( R = 2.8 \) and \( R = 4.2 \), at three different values of the total stress per particle \( \tilde{p}_1 \). For the smaller \( R \), the agreement is not bad, but the prediction of Eq. (S2) noticeably curves away from the data at both the high and low ends; for the larger \( R \), the agreement seems quite good.

To make a quantitative test, we can compute the chi squared per degree of freedom, \( \chi^2/\nu \), of the fit of the data to the predicted curve of Eq. (S2), and then compare that to the \( \chi^2/\nu \) of the fit described in the main article,

\[ \ln \left( \frac{P_1}{P_2} \right) = c + \Delta \alpha \Gamma_R + \Delta \lambda \Gamma_R^2, \]  
\[ \chi^2/\nu \]

where \( \Delta \alpha \) and \( \Delta \lambda \) are taken to have the same values for all cluster sizes \( R \), and \( c_R \) varies with \( R \). In Fig. S2 we plot \( \chi^2/\nu \) for these two fits as a function of cluster radius \( R \), for three different values of \( \tilde{p}_1 \). We see that the fit of Eq. (S6), based on the Boltzmann-like distribution, generally gives a better fit with smaller \( \chi^2/\nu \) than does the Gaussian distribution of Eq. (S2).

For even clearer evidence, we turn to the joint distribution of stress and force-tile area, \( \mathcal{P}(\Gamma_R, A_R) \), and make a similar Gaussian assumption for this joint distribution,

\[ \mathcal{P}(\Gamma_R, A_R) = \exp \left[ -\frac{1}{2} (\delta \Gamma_R, \delta A_R) \cdot \Sigma^{-1} \cdot (\delta \Gamma_R, \delta A_R) \right] \]

where \( \delta \Gamma_R \equiv \Gamma_R - \langle \Gamma_R \rangle \), \( \delta A_R \equiv A_R - \langle A_R \rangle \), \( \Sigma \) is the covariance matrix,

\[ \Sigma = \begin{pmatrix} \langle \delta \Gamma_R^2 \rangle & \langle \delta \Gamma_R \delta A_R \rangle \\ \langle \delta \Gamma_R \delta A_R \rangle & \langle \delta A_R^2 \rangle \end{pmatrix}, \]

and \( |\Sigma| \) is the determinant of \( \Sigma \). Computing the log ratio using this distribution then gives the parabolic surface,

\[ \ln \left( \frac{P_1}{P_2} \right) = c + a \Gamma_R + b \Gamma_R^2 + u A_R + v A_R^2 + \nu \Gamma_R A_R, \]
\[ c = \frac{1}{2} \ln \left( \frac{|\Sigma_2|}{|\Sigma_1|} \right) + \frac{\langle \delta A_R^2 \rangle_1}{2|\Sigma_2|} (\langle \Gamma_R \rangle_2)^2 - \frac{\langle \delta A_R^2 \rangle_1}{2|\Sigma_1|} (\langle \Gamma_R \rangle_1)^2 \]
\[ + \frac{\langle \delta \Gamma_R^2 \rangle_2}{2|\Sigma_2|} (\langle A_R \rangle_2)^2 - \frac{\langle \delta \Gamma_R^2 \rangle_1}{2|\Sigma_1|} (\langle A_R \rangle_1)^2 \]
\[ - \frac{\langle \delta \Gamma_R \delta A_R \rangle_2 (\langle A_R \rangle_2)^2}{|\Sigma_2|} + \frac{\langle \delta \Gamma_R \delta A_R \rangle_1 (\langle A_R \rangle_1)^2}{|\Sigma_1|} \]  
(S10)

\[ a = \frac{\langle \delta A_R^2 \rangle_1}{|\Sigma_1|} - \frac{\langle \delta A_R^2 \rangle_2}{|\Sigma_2|} \]
\[ + \frac{\langle \delta \Gamma_R \delta A_R \rangle_2 (\langle A_R \rangle_2)^2}{|\Sigma_2|} + \frac{\langle \delta \Gamma_R \delta A_R \rangle_1 (\langle A_R \rangle_1)^2}{|\Sigma_1|} \]  
(S11)

\[ b = \frac{\langle \delta A_R^2 \rangle_2}{2|\Sigma_2|} - \frac{\langle \delta A_R^2 \rangle_1}{2|\Sigma_1|} \]
\[ + \frac{\langle \delta \Gamma_R \delta A_R \rangle_2 (\langle A_R \rangle_2)^2}{|\Sigma_2|} - \frac{\langle \delta \Gamma_R \delta A_R \rangle_1 (\langle A_R \rangle_1)^2}{|\Sigma_1|} \]  
(S12)

\[ u = \frac{\langle \delta \Gamma_R^2 \rangle_1}{|\Sigma_1|} - \frac{\langle \delta \Gamma_R^2 \rangle_2}{|\Sigma_2|} \]
\[ + \frac{\langle \delta \Gamma_R \delta A_R \rangle_2 (\langle A_R \rangle_2)^2}{|\Sigma_2|} - \frac{\langle \delta \Gamma_R \delta A_R \rangle_1 (\langle A_R \rangle_1)^2}{|\Sigma_1|} \]  
(S13)

\[ v = \frac{\langle \delta \Gamma_R^2 \rangle_2}{2|\Sigma_2|} - \frac{\langle \delta \Gamma_R^2 \rangle_1}{2|\Sigma_1|} \]  
(S14)

\[ w = \frac{\langle \delta \Gamma_R \delta A_R \rangle_1}{|\Sigma_1|} - \frac{\langle \delta \Gamma_R \delta A_R \rangle_2}{|\Sigma_2|} \]  
(S15)

In Fig. S3 we plot our data for the log ratio, together with the predicted surface of Eq. (S9), vs \( \Gamma_R \) and \( A_R \). We show results for a cluster radius \( R = 4.2 \) at \( \tilde{p}_1 = 0.00078 \), and for \( R = 2.8 \) at \( \tilde{p}_1 = 0.00225 \). Our plots, with the perspective oriented to an edge-on view of the data, clearly show the data lies on a flat plane, while the parabolic surface predicted by Eq. (S9) noticeably curves away on the scale of the data.

We can compare this Gaussian assumption against the prediction from Boltzmann-like distribution, as described in the main article,

\[ \ln \left( \frac{P_1}{P_2} \right) = c_R + \Delta \Gamma_R + \Delta \lambda A_R \]  
(S16)

where \( \Delta \alpha \) and \( \Delta \lambda \) are taken to have the same values for all cluster sizes \( R \), and \( c_R \) varies with \( R \). In Fig. S4 we plot the \( \chi^2/\nu \) of the fit of the data to both the Gaussian and Boltzmann predicted surfaces vs cluster radius \( R \). We show results for three different values of \( \tilde{p}_1 \). We see that the Boltzmann form gives a significantly better fit than does the Gaussian assumption for all \( R \) and all \( \tilde{p}_1 \).

Our results here thus give clear evidence that the stress distributions of our jammed clusters are better described by the Boltzmann-like distribution of the maximum entropy hypothesis than they are described by a simple Gaussian.
[14] This is in contrast to Ref. [7] where, for non-periodic clusters within the ordered triangular force-network ensemble, it was found that both $\alpha$ and $\lambda$ (their $\gamma$) vary with the cluster size, and that $\lambda$ is vanishing as the cluster size increases. This difference may be due to the simplifications of the force-network ensemble as compared to the more realistic jammed packings studied here.
[15] See our Supplemental Material for comparison between our results here vs the assumption that the stress is described by a simple Gaussian distribution.