4. Gauss's Law

S. G. Rajeev

January 27, 2009

1 Statement of Gauss's Law

> The electric flux through any closed surface is proportional to the total charge contained inside it.

In other words

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}.$$

2 Gauss's Law Implies Coulomb's Law

We can derive Coulomb's law from Gauss's law. This is essentialy the reverse of the argument we gave in the last lecture.

Consider a point-charge Q at some point. The flux through a sphere of radius r with center at this charge is easy to calculate because the electric field is normal to the sphere. This follows from the symmetry of the sphere. There is no other direction it can point in: it has to either point outward or inward.

$$\mathbf{E} = \pm|\mathbf{E}|\hat{\mathbf{r}}. \tag{1}$$

Therefore the flux through a small area of the surface is $\mathbf{E} \cdot d\mathbf{A} = \pm|\mathbf{E}||d\mathbf{A}|$ since the area points radially outward.

The electric field has constant magnitude on the surface of the sphere. Thus $\oint \mathbf{E} \cdot d\mathbf{A} = \pm|\mathbf{E}| \oint |d\mathbf{A}| = \pm 4\pi r^2 |\mathbf{E}|$. Setting this equal to $\frac{Q}{\epsilon_0}$ gives $|\mathbf{E}| = \pm\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$. Putting into the above formula 1 for $\mathbf{E}$ we get

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.$$

The force on a charge at the point $\mathbf{r}$ will be $q\mathbf{E}$, so that

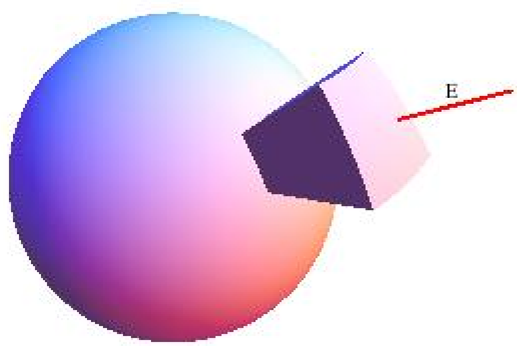
$$\mathbf{F} = k\frac{qQ}{r^2}\hat{\mathbf{r}}$$

where $k = \frac{1}{4\pi\epsilon_0}$. This is Coulomb's Law.

3 Gauss's Law Follows From Coulomb's Law

Gauss's law is actually a consequence of Coulomb's Law and the law that the forces due to many charges is the sum of the force due to each charge. For the case of a single charge and a surface which is a sphere centered on it, we proved it in the last class. To prove it for more general surfaces and charges, we have to think of a way of breaking them up into this situation.

First of all, consider a single point charge surrounded by a sphere of radius one. Now cut out a small piece of area $d\Omega$ of the sphere and pull it out: the sides are parallel to the electric field so carry no flux. The face is part of a sphere of larger radius r. The flux through this is the same as that through the piece that was cut out: the increase in area of the face due to the larger radius $(r^2 d\Omega)$ is exactly compensates for the decrease in electric field $k\frac{Q}{r^2}$so that the flux is $kQd\Omega$.



We can pull out many such small pieces to make a patchwork that fits any surface surrounding the charge. Each piece will have the same flux as the corresponding part of the unit sphere surrounding the charge. Thus the total flux through any surface surrounding the point charge is the same as for a unit sphere

$$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi kQ.$$

Now suppose there are two point charges inside this surface. We know that the electric field is the sum of the electric fields due to each charge. The flux will be the sum of the flux of electric fields produced by each charge:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint \mathbf{E}_1 \cdot d\mathbf{A} + \oint \mathbf{E}_2 \cdot d\mathbf{A} = 4\pi k Q_1 + 4\pi k Q_2 = 4\pi k[Q_1 + Q_2].$$

If there are many charges, we repeat the argument for each one to get

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint \mathbf{E}_1 \cdot d\mathbf{A} + \oint \mathbf{E}_2 \cdot d\mathbf{A} + \cdots = 4\pi k Q_1 + 4\pi k Q_2 + \cdots = 4\pi k[Q_1 + Q_2 + \cdots] = 4\pi Q.$$

where Q is the total charge inside the surface.

The tricky part of this argument is to show that any smooth surface can be approximated by small pieces of spheres centered at a point inside, each patch having a different radius. To really prove this satisfactorily uses ideas from the field of mathematics known as *measure theory.* But we don't go that deep into the proof.

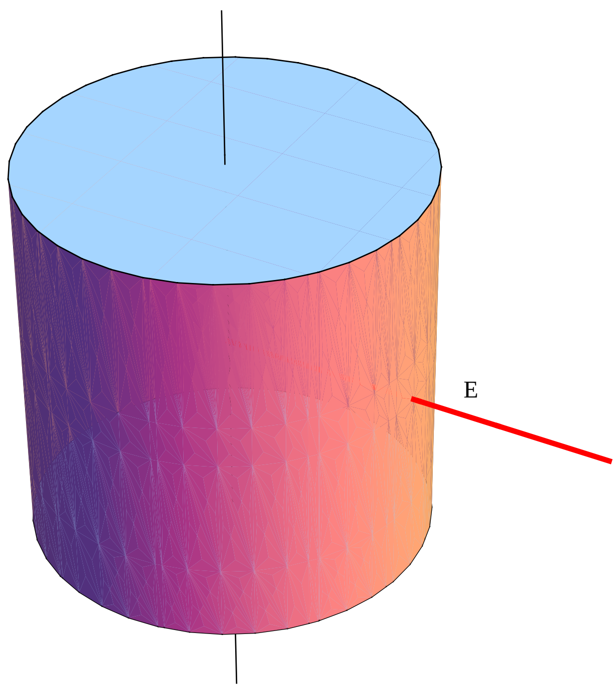
The advantage of the way we cut up a surface into spheres is that radial vector is always normal to the surface (for the part of the sphere) or tangential to it (the sides).

4 The Electric Field of a Line of Charges

Imagine now a large number of charges, each of small magnitude, that are arranged with constant density along a line. What is the electric field produced by them? If we take a piece of the line of length L, the electric charge on it will be some constant σ (the charge per unit length) times L

$$Q = \sigma L.$$

Imagine that the charges are along the vertical axis. The electric field at any point has to be directed along the horizontal line to the charges, by symmetry.Now imagine a cylinder passing through your point, whose axis is along the charges. The normal to the cylinder points along the electric field at any point. Also the electric field has the same magnitude at all points on the cylinder.



So the flux of the Electric field is

$$EA = 4\pi kQ.$$

The area of the cylinder is

$$A = 2\pi rL$$

Remembering that the charge is also proportional to L, we get

$$2\pi rLE = 4\pi k\sigma L$$

so that the magnitude of the eletric field is

$$E = k\frac{2\sigma}{r}.$$

It decays like $\frac{1}{r}$ instead of $\frac{1}{r^2}$.as for a point charge.

5 The Electric Field of a Plane of Charges

Suppose we have a plane carrying a constant charge density of σ per unit area. The electric field it produces must be pointed normal to the plane. (No other direction is special.)

This time we imagine a surface that is a box whose sides are planes also, two of them parallel to the charges each with area A. The electric field is normal to these planes. The electric flux is EA on each side and the charge inside is σA. Thus

$$2EA = 4\pi k\sigma A$$

Thus the electric field is a constant:

$$E = 2\pi k\sigma$$

6 Spherically Symmetric Charge Distribution

Suppose the charge density ρ depends only on the distance r from some point. The electric field has to poinr radially. The flux through a sphere of radius r is $E(r)4\pi r^2$.

If the total charge inside this sphere is $Q(r)$ Gauss's law gives

$$E(r)4\pi r^2 = \frac{Q(r)}{\epsilon_0}$$

.

$$Q(r) = 4\pi \int_0^r \rho(r)r^2 dr$$

Thus the electric field at a distance r from the center is the same as if all the charge inside a sphere of radius r were concentrated at the center:

$$E(r) = k\frac{Q(r)}{r^2}$$

The total charge inside is given by an integral

$$Q(r) = 4\pi \int_0^r \rho(r)r^2 dr.$$

As an example, suppose the density is a constant upto a distance R and is zero for larger distances. Then for $r < R$,

$$Q(r) = \frac{4\pi}{3}r^3\rho$$

You can see this by doing an integral; but it is also just the density times the volume of a spher of radius r.

Thus the electric field for $r < R$ is

$$E(r) = \frac{1}{3\epsilon_0}\rho r.$$

It grows with distance.

But if $r > R$ the total charge is a constant:

$$Q(R) = \frac{4\pi}{3}R^3\rho.$$

Thus the electric field decreases with distance:

$$E(r) = k\frac{Q(R)}{r^2}.$$

It is as if all the charge is concentrated at the origin. This is a special property of spherical charge distributions.