5. Potential Energy

S. G. Rajeev

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We know now the force on a charged particle. Next we will understand its energy.

1 Conservative Forces

Recall that if you exert a constant force $\mathbf{F}$ on a particle and it is displaced by $\mathbf{a}$, the work you have done is $\mathbf{F} \cdot \mathbf{a}$. The energy of the particle increases by that amount. If the force and displacement are in the same direction, this is positive; if they are in opposite directions, the work done is negative and the energy particle actually decreases (increases by a negative amount). If the displacement is perpendicular to the force, no work is done, no matter how large the force; and the energy of the particle does not change.

Now suppose that the particle is in a force field, that varies from point to point in space. It can still be thought of as constant for a small displacement $d\mathbf{l}$ so that the work done is $dW = \mathbf{F} \cdot d\mathbf{l}$. If the particle moves along a path connecting $\mathbf{r}_1$ to $\mathbf{r}_2$ the work done can be found by breaking up the path into small steps and adding up the work done in each step. This is the integral

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{l}$$

In general this will depend on the path: some paths might get you from $\mathbf{r}_1$ to $\mathbf{r}_2$ with less energy than others. But there are some force fields for which the work done only depends on the endpoints: it doesn't matter how long or short the path connecting them is, the work done is the same. Such forces are called conservative forces.

An example of this gravity. If you move a weight from one point to another near the surface of the Earth, the work you did only depends on the height difference between $\mathbf{r}_1$ and $\mathbf{r}_2$. If they are at the same height you have done no work, no matter how much you sweated while moving the weight.

It turns out that another example of a conservative force is the Coulomb force between static charges. This should not surprise you: the Coulomb force law and Newton's law of gravity are so similar that if one is conservative the other had to be also. Magnetic forces are not conservative because they depend on the velocity, not just the position of a particle. But that is not our concern right now.

2 The Potential Energy of a Conservative Force

Consider a force that is conservative, so that the work done depends only on the endpoints and not on the path we chose to connect them. Suppose we went from $\mathbf{r_1}$ to $\mathbf{r_2}$ and then onto another point $\mathbf{r}_3$. Then the total work done is the sum of the integrals over the two:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_3} \mathbf{F} \cdot d\mathbf{l} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{l} + \int_{\mathbf{r}_2}^{\mathbf{r}_3} \mathbf{F} \cdot d\mathbf{l} \tag{1}$$

Because it is conservative, the l.h.s. can only depend on $\mathbf{r}_1$ and $\mathbf{r}_3$while the first term on the right depends on $\mathbf{r}_1$ and $\mathbf{r}_2$and the second term on the right depends on $\mathbf{r}_2$ and $\mathbf{r}_3$. Thus the dependence on $\mathbf{r}_2$must cancel between the two terms on the r.h.s. This can happen only if there is a function $V(\mathbf{r})$ such that

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{l} = U(\mathbf{r}_1) - U(\mathbf{r}_2).$$

Write out the sum of the two terms on the r.h.s. of (1) and check that the dependence on $\mathbf{r}_2$ does indeed cancel. This function is called the potential energy of the force.

If you knew $V(\mathbf{r})$, you could determine the force by differentiation. Consider a very small step from $\mathbf{r}$ to a nearby point $\mathbf{r} + d\mathbf{l}$:

$$\mathbf{F} \cdot d\mathbf{l} = U(\mathbf{r}) - U(\mathbf{r} + d\mathbf{l})$$

This is saying that the force is the negative of the derivative of the potential energy. If a force satisfies this condition at any point, it is conservative.

3 The Coulomb Potential

We will show now that the Coulomb force is conservative.

Suppose we have a charge Q at the origin. The force on a charge q at some point $\mathbf{r}$ is

$$\mathbf{F}(\mathbf{r}) = k\frac{qQ}{r^2}\hat{\mathbf{r}}.$$

This is just Coulomb's law. We must come up with a function $U(\mathbf{r})$ such that

$$k\frac{qQ}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{l} = U(\mathbf{r}) - U(\mathbf{r} + d\mathbf{l})$$

for any small step $d\mathbf{l}$. If you know a bit of calculus you can guess the answer: we asking for the integral of the force. We will show that the answer is

$$U(\mathbf{r}) = k\frac{qQ}{r}.$$

To see that this works, we must calculate the change in $U(\mathbf{r})$ under a small step.

3.1 Some Math

Now, recall that $r^2 = \mathbf{r}\cdot\mathbf{r}$: the square of the length of a vector is its dot product with itself. So

$$|\mathbf{r} + d\mathbf{l}|^2 = (\mathbf{r} + d\mathbf{l}) \cdot (\mathbf{r} + d\mathbf{l}) = r^2 + 2\mathbf{r} \cdot d\mathbf{l}$$

We ignore the square of the small quantity.

Now,

$$|\mathbf{r} + d\mathbf{l}|^2 = r^2 + 2\mathbf{r} \cdot d\mathbf{l} = r^2 \left(1 + \frac{2}{r^2}\mathbf{r} \cdot d\mathbf{l}\right)$$

This tells us how $r^2 = \mathbf{r}\cdot\mathbf{r}$ changes. But what we need is to know how r^{-1} changes. But $r^{-1} = (r^2)^{-\frac{1}{2}}$

Thus

$$\frac{1}{|\mathbf{r} + d\mathbf{l}|} = \left(r^2 + 2\mathbf{r} \cdot d\mathbf{l}\right)^{-\frac{1}{2}} = \frac{1}{r}\left(1 + \frac{2}{r^2}\mathbf{r} \cdot d\mathbf{l}\right)^{-\frac{1}{2}}.$$

Now remember that for a small quantity

$$(1 + \epsilon)^n = 1 + n\epsilon.$$

This is a basic fact of algebra. Use your calculator to verify it for different small values of ϵ. We need it for $n = -\frac{1}{2}$.

$$\left(1 + \frac{2}{r^2}\mathbf{r} \cdot d\mathbf{l}\right)^{-\frac{1}{2}} = 1 - \frac{1}{r^2}\mathbf{r} \cdot d\mathbf{l}$$

Dividing by r,

$$\frac{1}{|\mathbf{r} + d\mathbf{l}|} = \frac{1}{r} - \frac{1}{r^2}\frac{\mathbf{r}}{r} \cdot d\mathbf{l}$$

Remember that

$$\frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.$$

We get

$$\frac{1}{|\mathbf{r} + d\mathbf{l}|} - \frac{1}{r} = -\frac{1}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{l}$$

Rearranging the sides

$$\frac{1}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{l} = \frac{1}{r} - \frac{1}{|\mathbf{r} + d\mathbf{l}|}$$

3.2 The Last Step

We are almost there. Just multiply both sides by kqQ to get

$$k\frac{qQ}{r^2}\hat{\mathbf{r}}\cdot d\mathbf{l} = k\frac{qQ}{r} - k\frac{qQ}{|\mathbf{r}+d\mathbf{l}|}$$

which is what we had to prove.

Thus we showed that the Coulomb potential is a conservative force. And we even found the potential energy of the Coulomb force. This is a big step forward.

4 The Electric Potential

Recall that the force on a particle is proportional to its charge. The force on a particle divided by its charge is the electric field. The electric field due to a point charge is

$$\mathbf{E} = k\frac{qQ}{r^2}\hat{\mathbf{r}}.$$

So it woud make sense to think of the potential energy divided by the electric charge q. This is called the electrostatic potential.

Electric Field =Force divided by charge

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

Electric Potential= Potential energy divided by charge

$$V(\mathbf{r}) = \frac{U(\mathbf{r})}{q}$$

Dividing by q,

$$k\frac{Q}{r^2}\hat{\mathbf{r}}\cdot d\mathbf{l} = k\frac{Q}{r} - k\frac{Q}{|\mathbf{r}+d\mathbf{l}|}$$

The electric potential at $\mathbf{r}$ due to a point charge Q at the origin is,

$$V(\mathbf{r}) = k\frac{Q}{r}.$$

The eletric potential is k times the charge divided by the distance.

This is yet another way think of Coulomb's law. We had to work hard to arrive at the idea of a potential. The reward is a major simplification: in this form of the law, we have a scalar quantity V to deal with instead of the electric field $\mathbf{E}$, which is a vector.

5 The Electric Potential of Many Charges

The electric field produced by two charges is the sum of the electric fields due to each one. The same is true of the eletric potential. But the potential being a scalar, it is much easier to add them. If there are charges Q_1 and Q_2at positions $\mathbf{r}_1$and $\mathbf{r}_2$respectively, the potential is

$$V(\mathbf{r}) = k\frac{Q_1}{|\mathbf{r}-\mathbf{r}_1|} + k\frac{Q_2}{|\mathbf{r}-\mathbf{r}_2|}$$

If we have many charges at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{3,} \cdots$ we get

$$V(\mathbf{r}) = k\frac{Q_1}{|\mathbf{r}-\mathbf{r}_1|} + k\frac{Q_2}{|\mathbf{r}-\mathbf{r}_2|} + k\frac{Q_3}{|\mathbf{r}-\mathbf{r}_3|} + \cdots$$

and so on. This way we can find the electric potential of any distribution of charges. We will look at some examples in the next class.