

6. Electrical Potential

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1 Partial Derivatives

If you have a function of one variable $f(x)$, its derivative measures how much it changes under a small change of x :

$$\frac{df}{dx} = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta}.$$

Thus

$$\frac{dx^2}{dx} = 2x, \quad \frac{d}{dx} (3x + \sqrt{x}) = 3 + \frac{1}{2\sqrt{x}}$$

etc.

But many functions depend on more than one variable. For example, the potential energy $U(\mathbf{r})$ of a particle depends on its position, which is specified by three co-ordinates $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the three co-ordinate axes. Thus we can think of U as a function of three variables $U(x, y, z)$. The force acting on a particle is the negative derivative of the potential (for conservative forces). The component of the force in the x -direction is the derivative of U with respect to x keeping y, z fixed.

This is called the partial derivative:

$$\frac{\partial U}{\partial x} = \lim_{\delta \rightarrow 0} \frac{U(x + \delta, y, z) - U(x, y, z)}{\delta}.$$

The little curly ∂ is also pronounced ‘d’: it is just an old way of writing cursive ‘d’.

There are of course similar partial derivatives with respect to y, z :

$$\frac{\partial U}{\partial y} = \lim_{\delta \rightarrow 0} \frac{U(x, y + \delta, z) - U(x, y, z)}{\delta}, \quad \frac{\partial U}{\partial z} = \lim_{\delta \rightarrow 0} \frac{U(x, y, z + \delta) - U(x, y, z)}{\delta}.$$

2 Gradient

The three partial derivatives are the components of a vector called the ‘gradient’ of U :

$$\nabla U = \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k}.$$

The symbol ∇ is pronounced ‘grad’, short for gradient. If we change \mathbf{r} to $\mathbf{r} + d\mathbf{l}$ for a small vector $d\mathbf{l}$, then

$$d\mathbf{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\nabla U \cdot d\mathbf{l} = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz.$$

In other words

$$\nabla U \cdot d\mathbf{l} = U(\mathbf{r} + d\mathbf{l}) - U(\mathbf{r})$$

2.1 Examples

$$U(x, y, z) = x^2 y \Rightarrow \frac{\partial U}{\partial x} = 2xy, \quad \frac{\partial U}{\partial y} = x^2, \quad \frac{\partial U}{\partial z} = 0, \quad \nabla U = 2xy\mathbf{i} + x^2\mathbf{j}.$$

We showed in the last class that

$$\nabla \frac{1}{r} = -\frac{1}{r^2} \hat{\mathbf{r}}.$$

Here is another way to do prove it. Start with

$$r^2 = (x^2 + y^2 + z^2)$$

so that

$$\frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}.$$

Then

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = \left(-\frac{1}{2} \right) 2x (x^2 + y^2 + z^2)^{-\frac{3}{2}} = -x r^{-\frac{3}{2}}$$

where we treat y, z as if they are constants. That is,

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{x}{r}.$$

In exactly the same way we can see that

$$\frac{\partial}{\partial y} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{y}{r}, \quad \frac{\partial}{\partial z} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{z}{r}.$$

Putting the three components together

$$\nabla \frac{1}{r} = -\frac{1}{r^2} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r} = -\frac{1}{r^2} \hat{\mathbf{r}}.$$

A little knowledge of calculus goes a long way. A lot of the difficulties that people have with learning physics is that they haven’t taken the time to learn a bit of the mathematics needed. This is true at every level, from High School teachers to research physicists. But once you learn the math (which is actually easier than learning physics) you can circles around the people who haven’t.

3 The Electric Potential

The main point of the last class was that for a conservative force,

$$\mathbf{F} = -\nabla U.$$

The electrical force is conservative. It is also proportional to the charge q of the particle sitting at the point \mathbf{r} . If we divide by this charge both sides, we get

$$\mathbf{E} = -\nabla V$$

where $V(\mathbf{r}) = \frac{U(\mathbf{r})}{q}$ is the potential energy per unit charge of a particle at the point \mathbf{r} . It is called the electric potential. From Coulomb's law, the electrical potential created at a point \mathbf{r} by a charge Q at the origin is

$$k \frac{Q}{r}.$$

If you have charges Q_1, Q_2, \dots at points $\mathbf{r}_1, \mathbf{r}_2 \dots$ the total potential created at point \mathbf{r} by them is

$$k \left[\frac{Q_1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{|\mathbf{r} - \mathbf{r}_2|} + \dots \right]$$

We can use this to calculate the potential due to many distributions of charges.

4 The Unit of Electric Potential

The unit of force is *Newton* and the unit of charge is *Coulomb*. The unit of electric field is thus NC^{-1} . The unit of work (or energy) is *Joules* , which is the same as Nm . Thus the unit of potential energy is $JC^{-1} = NmC^{-1}$. This has another name *Volt*. One Coulomb is such a large amount of energy that you are likely to encounter only a few μC in practice. But it turns out that one Volt is a common potential.

A typical alkaline cell (like the AAA battery that powers my microphone) produces an electrical potential difference of 1.5 Volts. The different sizes AAA, AA, A, B, C, D, DD etc. all produce the same potential, but simply last longer.



(What other consumer product is sold in sizes AAA, AA, A, B, C, D, DD etc.? Hint: The answer has nothing to do with physics.)

5 The Electric Potential in the Hydrogen Atom

The charge on an electron is a tiny fraction of a Coulomb $-1.6 \times 10^{-19}C$. Yet the potential seen by an electron is a few Volts: the Volt is one of the few units that work just as well at the atomic scale as in every day life.

Let us calculate the electrical potential in a hydrogen atom.

The charge of its nucleus is $1.6 \times 10^{-19}C$. The size of the atom (radius of the electron's orbit) is one Bohr radius; i.e., $r = 6 \times 10^{-11}m$. Thus

$$V(r) = k \frac{Q}{r} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{1}{4\pi(8.9 \times 10^{-12})} \frac{1.6 \times 10^{-19}}{5.3 \times 10^{-11}} = \frac{1.6}{4 \times 3.14 \times 8.9 \times 5.3} 10^4 = 27V$$

Other atoms are bigger and their outermost electron (called the valence electron) is what takes part in chemical reactions. At those distances the electrical potential is a few volts. That is why batteries based on chemical reactions produce a few volts.

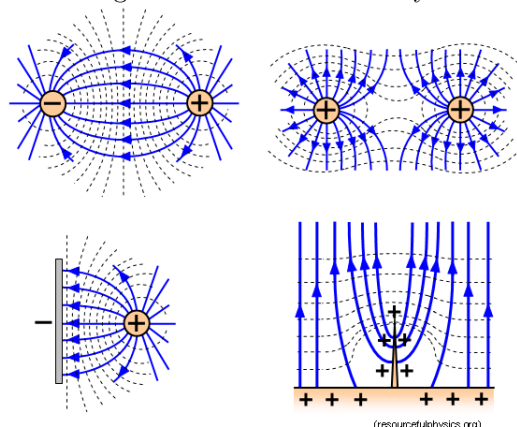
If you want larger potentials, you can string several batteries in series. But for much larger potentials, you need special equipment like a van de Graf generator, which can create about a million volts.

6 Equipotential Surfaces

A surface on which the potential is a constant is called an *equipotential surface*. For gravity this are points at the same height; more precisely (assuming the Earth is a sphere) at the same distance from the center of the Earth.

An equipotential surface of a positive point charge is a sphere: as the potential decreases, the radius of the sphere grows.

The electric field is always normal the surface: it points in the direction of decreasing potential. Thus for a point positive charge the field points outward. Plotting equipotential surfaces and electrical field lines is a good way of understanding the electrical field of a system of charges.



In this figure, the thick blue lines are the electric field lines. The dashed lines are the equipotential surfaces.

Notice that they are always perpendicular.

7 Potential of a Dipole

An electric dipole are two equal and opposite charges located close together. Their electrical potential is

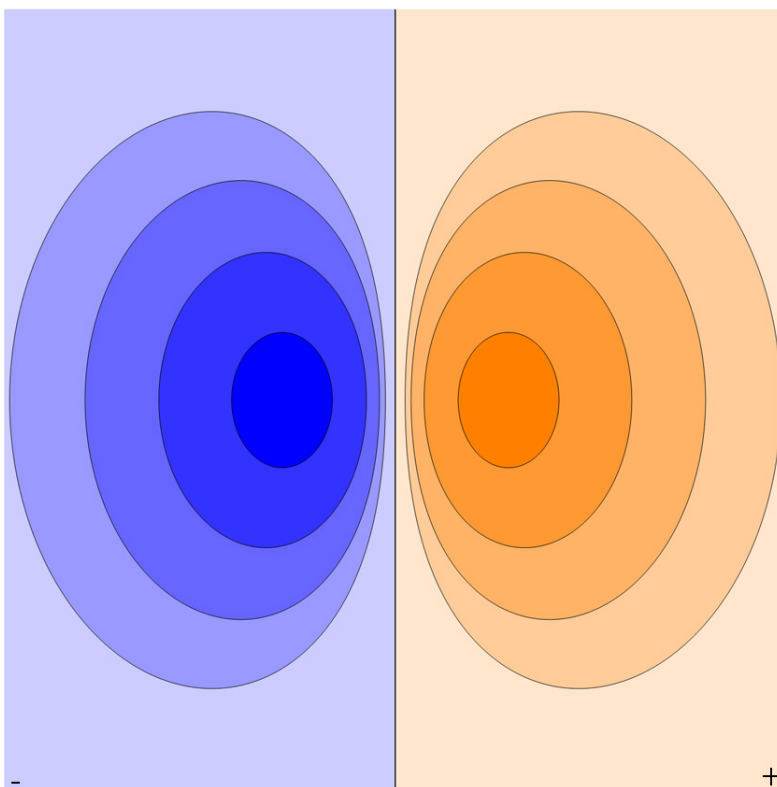
$$\frac{1}{4\pi\epsilon_0} \frac{Q}{|\mathbf{r}|} - \frac{1}{4\pi\epsilon_0} \frac{Q}{|\mathbf{r} + \mathbf{a}|}.$$

At distances large compared to $|\mathbf{a}|$, we can treat the displacement between them as infinitesimal. Then this becomes (much like the calculation earlier of the gradient of $\frac{1}{r}$)

$$-\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} Q \mathbf{a} \cdot \hat{\mathbf{r}}.$$

The quantity $\mathbf{p} = Q\mathbf{a}$ is called the dipole moment. Thus the electrical potential of a dipole is

$$V(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}.$$



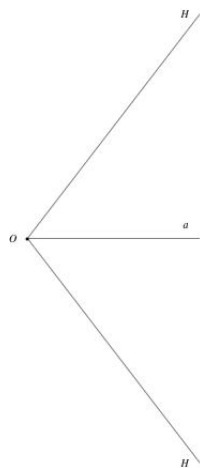
Shown above (is a cross-section) of the equipotential surfaces of a dipole oriented along the horizontal axis.

7.1 The dipole moment of a water molecule

Water is H_2O . The two hydrogen molecules lie along the vertices of a triangle with the Oxygen at the third vertex. The angle between the H molecules is 105 degrees. The overall charge of the molecule is zero, but the electrons of the hydrogen molecules shift their positions towards the Oxygen atom. By how much does the average position of such an electron shift? This can be determined by measuring the dipole moment of a water molecule, which turns out to be $p = 6.3 \times 10^{-30} Cm$. Dividing by the charge of an electron this is

$$a = \frac{p}{e} = 3.9 \times 10^{-11} m$$

This the distance between the average position of the positive charges and the average position of the negative charges.



The radius of the orbit of an electron in a hydrogen atom is $5.9 \times 10^{-11}m$. So the electrons in the hydrogen atom are shifted a fraction of this radius, in the water molecule.

7.2 Potential Energy of a Dipole

A dipole sitting in an electric field has an energy that is equal to the sum of the energies of each of its charges. This is

$$QV(\mathbf{r} + \mathbf{a}) - QV(\mathbf{r})$$

In the limit of small \mathbf{a} this becomes

$$-\mathbf{p} \cdot \mathbf{E}.$$

To minimize the energy, the dipole must point along the electric field. To dipoles located nearby each other will try to orient themselves so that the unlike charges are closer together.

An electric dipole is much like a magnet: there are in fact things called electrets which are the analogues of magnets: they produce electric fields instead of magnetic fields. They are basically a large collection of molecules, each with an electric dipole moment, all pointing the same way.

8 The Electrical Potential of a Charged Plane

Imagine a large sheet of constant electrical charge density σ . Gauss's law applied to a rectangular box surrounding it will give us the electric field E to be normal to the plane and of constant magnitude

$$E = \frac{\sigma}{\epsilon_0}$$

So the electric potential is

$$V(x) = \frac{\sigma}{\epsilon_0} x$$

where x is the axis perpendicular to the plane.

9 The Electrical Potential of a Ring of Charge

Imagine a ring of constant charge per unit length σ . It has radius R and the x -axis is normal to its plane, passing through its center. The potential at some point along this axis is the sum of those produced by point charges on the ring:

$$V(x) = \frac{1}{4\pi\epsilon_0} \sigma \int \frac{dl}{\sqrt{x^2 + R^2}} = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + R^2}}$$

The electric field on this axis is given by the negative derivative

$$E(x) = \frac{Q}{4\pi\epsilon_0} \frac{x}{(x^2 + R^2)^{\frac{3}{2}}}$$

The answers are more complicated away from the x -axis.