17. Maxwell's Equations

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All physics is based on experiments. There is no way we can deduce how nature works based on thought alone. People tried for centuries and got it wrong in every possible way.

In the case of electricity and magnetism, the fundamental laws of Gauss, Ampere and Faraday coalesced into an almost complete picture by the mid nineteenth century. But something was missing. The laws as known then led to contradictions. Maxwell suggested a modification to Ampere's law to remove this contradiction, thereby completing the laws of electricity and magnetism. These Maxwell's equations have held up to rigorous tests for a hundred and fifty years since then.

As a consequence of his modified Ampere's law, Maxwell could deduce that there were waves of electricity and magnetism, much like there are sound waves in air or water. The speed of these waves can be found in terms of ϵ_0 and μ_0.It turns out to be exactly that of light! Thus Maxwell was able to explain not just electricicty and magnetism but the more ancient phenomena of optics. In the rest of the course we will be studying the properties of light from this point of view.

The discovery of Maxwell's equations is the second greatest achievement of physics of all time: second only to Newton's discovery of the laws of motion and gravitation.

1 Maxwell's Discovery

Faraday showed that a magnetic field that changes with time will produce an electromotive force: an electric field. The law about producing magnetic fields is Ampere's law: the line integral of magnetic field around a curve is proportional to the total current passing through the surface it bounds.

In equations, we have

$$\oint \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}$$

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_S$$

Something is wrong here. How can a changing magnetic field produce an electric field but a changing electric field not produce a magnetic field? This goes against the usual law of action and reaction present everywhere else in physics.

There is also another problem. The left hand side of Ampere's law only depends on the boundary ∂S of the surface through which we are measuring the current. If we change the surface on the r.h.s. keeping its boundary fixed, it shouldn't change. But it is easy to produce situations where this is not true. (See the text for an example with capacitors). Maxwell noted that if we add a term to the r.h.s. of Ampere's law we can get out of this bind. The correct laws are

$$\oint \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}$$

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_S + \epsilon_0 \frac{d}{dt} \int_S \boldsymbol{E} \cdot d\mathbf{A}$$

That is, a time dependent electric field can produce a magnetic field. Besides being logically necessary this new term of Maxwell also makes the equations more symmetric: electric and magnetic fields can create each other if they are time-dependent. The second term is called the displacement current. (Displacement is an old word for electric field.)

Along with Gauss's law for eletric and magnetic fields, these constitute Maxwell's equations.

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

2   Waves in the Vacuum

Since electric and magnetic fields can create each other, they can exist even far from charges or currents. In the vacuum a time dependent electric field will produce a magnetic field and vice versa. If they depend periodically in time, it turns out that they must also depend periodically on space. These are typical of waves.

In a plane wave the field depends on the combination $kx - \omega t$. The quantity k is called the wave number; the period in space is called a wavelength and is given by $\lambda = \frac{2\pi}{k}$, just as the time period is given by $T = \frac{2\pi}{\omega}$. The ratio of the wavelength to the period is the speed at which the wave is moving:

$$c = \frac{\lambda}{T} = \frac{\omega}{k}.$$

These facts were already known before Maxwell, because physicists knew about waves on strings and sound waves- all of which behave this way. The speed of sound waves depend on the density and compressibility of air; on strings, it depends on the mass per unit length and the elasticity of the string.

In our case the speed can be calculates in terms of quantities ϵ_0 and μ_0. Maxwell's equations imply that the velocity is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

The actual argument is a bit beyond the scope of this course. But in an earlier lecture I explained how this quantity appears in the fields produced by a moving charge.

We can calculate this speed to be exactly

$$2.99792458 \times 10^8 ms^{-1}.$$

This is the speed of light in a vacuum. Maxwell must have fallen off his chair when he discovered this.

> Light is a wave of electric and magnetic fields.

3   Wavelength of Light

Different colors of light have different wavelengths (and hence frequencies). But all of them are quite small. Red light has wavelength 700nm while blue light has a shorter wavelength of 475nm. There are electromagnetic waves of all possible wavelengths. Some of them were known before, but were thought of as separate phenomena from light. The heat from a lamp is carried by e.m. waves in the infrared (below red in frequency) of wavelength about $10\mu m$. Radio waves can have wavelength of several meters: microwaves have wavelength of a few cm. The longest wavelenghth ever detected is a few hundred kilometers. The shortest are gamma rays, with a wavelength of about a femtometer: $10^{-15}m$.

There is no theoretical limit on how small or how large the wavelength of e.m. waves can be. Visible light is a tiny sliver of this allowed spectrum: it is the region where the atmosphere of the Earth is transparent. Our eyes evolved to see in that tiny window.

Portions of the e.m. spectrum are owned by the public: they are used to transmit TV and radio and wireless internet. They are very valuable and are auctioned off by the FCC.

4   Relation of E and B in a Wave

Because they create each other through their time dependence, the magnitudes of the electric and magnetic fields ina wave are related to each other:

$$E = cB.$$

Moreover, they are always pointed perpendicular to each other and to the direction in which the wave is moving.

$$\mathbf{k} \perp \mathbf{E} \perp \mathbf{B}.$$

These are all consequences of Maxwell's equations. But to derive them we need calculus beyond what you might be comfortable with.

The simplest kind of e.m. wave is 'plane polarized'. In this case the wave propagates along some direction (say the z-axis). The electric field oscillates back and forth along some fixed direction (say x-axis) and the magnetic field oscillates along the direction $\perp$ to both (y-axis). The direction of the electric field is called the direction of polarization of the wave. There are two kinds of plane polarized light: electric field along x-axis or y-axis for waves moving along the z-axis.

Another possibility is 'circularly polarized' wave. Here the electric and magnetic fields rotate in the xy-plane while remaining $\perp$ to each other. There are two kinds of circular polarized light also: the field is rotating in the counter-clockwise or clockwise direction.

There are intermediate possibilities as well.

Each has its use.

An example are the 3D glasses you get at Disney World. The left filter allows only light of one polarization while the right eye recieves light of the other polarization. So we can project two different images on the same screen. Each eye sees a slightly different perspective, which looks to the brain like a 3D image.

The LCD screen on computers works by using polarization also. You can switch the direction of the light that is allowed to pass through by applying an electric field: it rotates the long thin molecules that make up the liquid crystal in the screen. With a source of light in the back this can be used to create images.

Simple laws of physics can be turned into myriad devices by clever engineers.