21. Relativity

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Today we will study one of the deepest and fascinating discoveries of physics, mainly due to Albert Einstein. I will not pretend that the theory of relativity has many everyday applications. It has uses, some good and some that are devastating. Every educated person must know something about it.

Back in the eighteenth century, Thomas Jefferson taught himself calculus and mechanics, because he felt that his own education was not complete until he learned Newton's work. Abraham Lincoln taught himself Euclid's Geometry, to train his mind to think logically. Knowledge may not always appear to be of immediate utility. It always prepares your mind to think. In any field of life that you end up pursuing, new and abstract ideas will arise. Part of the purpose of learning science and mathematics is to take that way of thinking to other situations. Both Jefferson and Lincoln started some of the best universities in the US because they were convinced of this.

Abstract ideas can affect the real world dramatically. Relativity has already changed the world. It explains that mass can be converted to energy. This in turn made possible the production of nuclear weapons. The balance of power in the world, and the lives of billions of people have already been affected by this. Just as knowledge of the chemistry of salts made gunpowder possible, knowledge of relativity and nuclear physics makes nuclear weapons possible. Knowledge is the ultimate source of power.

1 The speed of sound

Imagine that there is a source of sound that lasts for just a second, at some point. The sound waves will expand outward along a sphere of radius $r = vt$ that expands with time. v is the speed of sound in that medium (say air).

Someone moving towards the source at some speed u' will hear it at a distance $(v - u')t$. It is as though the speed of sound measured by the moving observer is the difference $(v - u')$. This is in fact true: we can measure the speed of sound using stationary and moving microphones and verify this fact.

So people thought at the end of the nineteenth century that the speed of light would behave the same way. If the light is moving in the same direction as the Earth (which is moving around the Sun) or in the opposite direction, it should have slightly different speeds. The speed of the Earth as it moves around the Sun is about $30km$ per second. That of light is much higher, about 300,000 km per second. Michelson invented a clever device called an interferometer that would be accurate enough to see this difference. He was going to measure the speed of the Earth as it goes around the Sun using his invention.

2 The speed of light

The result was astonishing. There was no change in the speed of light in the vacuum. It did not matter which way or how fast the observer was moving, the light always moved with the same speed in vacuum. Since then we have repeated the experiment with much greater accuracy and it still holds: light moved at the same speed in the vacuum no matter what.

Now, this should not have been a total surprise. Maxwell's theory of elecromagnetism actually predicts just this. But neither Maxwell nor any of the other physicists understood it clearly. They were not thinking as carefully as they should have been, because they thought that light was much like sound. They thought that light was a wave in a mysterious medium called ether that pervaded all of space, even the vacuum. The rules for adding velocities went back to the time of Newton and Galeleo. Why would a new-fangled theory like electromagnetism upset those rules? Why would light be so special, different from all other waves and particles?

3 Lorentz and Poincare'

Actually, some scientists had already noticed that light was different. They knew that Maxwell's equations implied that the rules of addition of velocities was not true for light. Poincare' in particular had found the right rule for the addition of velocities. (We will give the formula in a minute). For small velocities (compared to c) the new rule reduced to the old one. There are discrepancies as the velocities become close to c. If one of the velocities is actually equal to c, the new rules say that $u \oplus c = c$. That would explain Michelson's experiments.

But what these scientists did not realize is that such a change in the rule for addition of velocities require a profound change in all branches of physics. It does not just affect light, it affects also mechanics. Newton's laws of mechanics, which deals with familiar ideas like energy and momentum would have to change. Einstein, in 1905, discovered those changes. The new theory of mechanics that emerged is called the theory of relativity.

4 Addition of Velocities

What is the new law of addition of two velocities u and v? (We will only consider the simplest case where they are parallel. But they don't have to be in the same direction. So they can have different signs.) The combined velocity is given by (we will use a new symbol $\oplus$ to denote the new rule for addition)

$$u \oplus v = \frac{u+v}{1+\frac{uv}{c^2}}$$

where c is the speed of light in the vacuum. It is a strange formula. But once you arrive at it, you can check that it has all the required properties. If u and v are both small compared to c, the ratio $\frac{uv}{c^2}$ is very small and can be ignored. Then we get the usual rule for adding velocities. For example, if u and v are 1 km per second (about the speed of the fastest jet airplanes) $\frac{uv}{c^2} \approx 10^{-11}$:an effect that is so small that it would be hard to see except by very specialized equipment. But as $v \to c$ (keeping u fixed at a value less than c) $u \oplus v \to c$ as well. In fact the sum never exceeds c, although $u+v$ might get larger than c,the denominator will also grow so that the ratio becomes less than or equal to c.

In the case where $v = c$, the formula is tricky to use (you must think of it as a limit), but it turns out that the sum $u + v$ in that case is just c.

5 Kinetic Energy and Momentum

The above law of addition was already discovered by Lorentz and Poincare'. Nowadays, they are usually stated in terms of "Lorentz Transformations", but we won't go there.

But Einstein was unaware of this and rediscovered it on his own. He went further and found how to modify Newton's mechanics to fit with this new rule. The momentum of a particle of mass m and velocity $\mathbf{v}$ is

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}}.$$

Again, for small $|\mathbf{v}| << c$, this is the same as the usual rule of Newtonian physics. But for large $|\mathbf{v}|$ close to c it is substantially different. To keep the momentum finite, the speed of any particle with non-zero mass must be less than c.

Similarly the formula for energy is

$$E = \frac{mc^2}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}}.$$

Again, if $m \neq 0$, the speed must always be less than c in order that the particle have a meaningful (real) energy.

For snmall speeds, a bit of algebra shows that

$$E \approx mc^2 + \frac{1}{2}m|\mathbf{v}|^2$$

The second term is the familiar formula for kinetic energy. Since under normal situations (e.g., slow collisions) the mass of the particle does not change, the first term can be ignored. Its effect is only important if the mass can somehow change.

6 Nuclear Fusion

But in nuclear collisions, the mass can change. Two nuclei can combine together to form a new one with slightly less mass than the sum of the two. The results are stunning: this tiny difference in mass can lead to a hew amount of energy. This is because c^2 is a very big number. 1 kg of mass corresponds to 10^{17} Joules of energy: about equivalent to a hydrogen bomb, or about 20 million tons of TNT.

The Sun produces energy this way, through nuclear fusion. Every second, about four million tons of its matter is converted into energy. Since the Sun weighs 2×10^{30} kg it can still last several billion years.

7 The Twin Paradox

The Theory of Relativity changes the way we measure time and distance. For example, it says that the time duration between two events depends on who is measuring it.

If the observer is moving with a speed v, the time duration she measures will be longer than someone at rest:

$$\tau' = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

For normal speeds of a few km per sec this is very small.

Suppose a particle has a lifetime τ when it is at rest. After that time it decays (changes into) some other pair particles. An example of this is a particle called the muon which decays into an electron and another particle called a neutrino in about one microsecond. We can make muons live much longer by making them move in a circle at a speed very close to the speed of light. To do this we must accelerate them to a high energy. Also, we can see high energy muons produced by collisions in the upper atmosphere at the surface of the Earth. If their lifetimes had not got longer through the above formula of relativity they would have died out in less than a kilometer, less than the thickness of the atmosphere.

This effect is often called the twin paradox. There is no logical contradiction here. It is called a paradox only because it contradicts our guess based on every day experience. If one of a pair of identical twins stays still and another goes around in a circle at a high speed, the one at rest will die first. We see this every day in high energy physics labs: not with human twins but with small particles like the muons. It is a real effect without any doubt.

8 Doppler Shift

Although the speed of light does not depend on the speed of the observer, its frequency and wavelength do depend on the observer. So Mickelsson wouldhave been able to measure the velocity of the Earth around the Sun if he had measured instead the frequency of light from a distant star instead of its velocity.

Relativity predicts that the frequency increases if the observer is moving towards the source and decreases if he is moving away from the source. More precisely, if the component of velocity of the observer in the direction away from the source is v

$$f = f_0 \sqrt{\frac{c-v}{c+v}}.$$

Obviously we must have $v < c$ for this to make sense. Also, if $v > 0$ (the observer is moving away) the frequency decreases and if $v < 0$ (the observer is moving towards the source) it is greater than f_0 . Here f_0 is the 'natural' frequency of the source, as measured by someone at rest relative to the source. The wavelength and frequency are related by

$$c = \lambda f.$$

Since c does not change the wavelength must vary inversely with the frequency

$$\lambda = \lambda_0 \sqrt{\frac{c+v}{c-v}}.$$

In all this, we are considering light in the vacuum. Of course, the speed of light in a medium is changed by its refractive idnex and that is a completely different effect.

An application of the Doppler effect is to the radar speed gun that the police use to measure the speed of a car. The frequecny of the radar waves reflected from the car depends on its speed. Nowadays they use a laser reflected off the car also. (Why is the laser more effective?)

Doppler radar measures not only the position of a thunderstorm but the speed of the rain drops carried in it.

The ultimate application of the Doppler effect is to Cosmology. The light coming to us from atoms in faraway galaxies has lower frequency than when it is emitted. (We know the natural frequencies of atoms-they can be measured in the lab.) This means that those galaxies are moving away from us: the whole universe is expanding. The farther away two points in the universe are, the faster they are moving apart from each other!

9 General Theory of Relativity

Einstein's greatest achievement was a further refinement of the theory of relativity, which takes into account the effect of acceleration as well as velocity on space and time. He showed that this can explain gravity. But to do this you need a sophisticated branch of mathematics called Riemannian geometry. (Way beyond calculus!). In the years after 1915 when he proposed it, we have been able to verify it in some detail. It explains why the universe is expanding. It predicts the existence of black holes.

These are topics right at the forefront of current research in physics.