Electrostatics

Charge and Coulomb's law

Matter possesses various properties that cause forces of interaction between particles.

You already know one such property — the mass. The mass of a particle determines the magnitude of the gravitational force it exerts on other particles. Mass is always a positive scalar (i.e., real number).

Another such property is charge. Charge comes in two flavors: positive and negative. It is observed that like charges (charges of the same sign) repel, while opposite charges attract — this is unlike gravitation which is always attractive. The charge of a particle is a scalar quantity that can be positive or negative.

Charge is conserved

The total charge of all particles in a box remains constant in time, provided no particles pass through the walls bounding the box. Particles can be created and destroyed in the box; particles can combine and separate, but the total charge remains constant.
Charge is quantized.

It is found that all charges observed freely in nature are integer multiples of the charge on the electron. We denote the magnitude of this charge by \( e \).

The charge on the electron is \( -e \).

The charge on the proton is \( +e \).

Protons, and other strongly interacting particles (hadrons) are now believed to be made up of subparticles called quarks. The charge on a quark can be either \( \frac{1}{3} e \) or \( \frac{2}{3} e \). However, it is not possible to observe a free quark. Quarks always come bound in triplets or pairs whose total charge is an integer multiple of \( e \).

Although charges are quantized in units of \( e \), a fact of great importance in particle physics, in classical electromagnetic theory we will generally take the charge \( q \) of a particle to be any real number.
Coulomb's Force Law

Consider a particle 1 with charge \( q_1 \) at position \( \vec{r}_1 \) and a particle 2 with charge \( q_2 \) at position \( \vec{r}_2 \).

When both charges are at rest (velocities \( \vec{v}_1 = \vec{v}_2 = 0 \)) the force between the two charges is given by Coulomb's Law.

The force on particle 2 due to particle 1 is:

\[
\vec{F}_2 = k \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21}
\]

Here \( \vec{r}_{21} = \vec{r}_2 - \vec{r}_1 \) is the displacement from \( q_1 \) to \( q_2 \)

\[ r_{21} = |\vec{r}_{21}| \] distance from \( q_1 \) to \( q_2 \)

\[ \hat{r}_{21} = \frac{\vec{r}_{21}}{r_{21}} \] unit vector in direction from \( q_1 \) to \( q_2 \)

Direction of force \( \vec{F}_2 \) is along line connecting \( q_1 \) to \( q_2 \).

If \( q_1 q_2 > 0 \) (charges have same sign) then \( \vec{F}_2 \) is in direction \( +\hat{r}_{21} \) \( \Rightarrow \) points away from \( q_2 \)

\( \Rightarrow \) force is repulsive

If \( q_1 q_2 < 0 \) (charges have opposite signs) then \( \vec{F}_2 \) is in direction \( -\hat{r}_{21} \) \( \Rightarrow \) points into \( q_1 \)

\( \Rightarrow \) force is attractive
$|\vec{F}_1| = \frac{1}{r_{12}^2}$ decreases as square of distance between charges "inverse square" force law.

Similarly, force on $q_1$ due to $q_2$ is:

$$\vec{F}_1 = \frac{k \cdot q_1 \cdot q_2}{r_{12}^2} \cdot \hat{r}_{12} = -\vec{F}_2 \quad \text{since} \quad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = -\vec{r}_2$$

$$\Rightarrow |\vec{r}_{12}| = |\vec{r}_2| \quad \text{and} \quad \hat{r}_{12} = -\hat{r}_2$$

So force on $q_1$ from $q_2$ is equal but opposite to force on $q_2$ from $q_1$ (Newton's 3rd law)

In the above Coulomb's law, $k$ is a universal constant of nature. It does NOT depend on the magnitudes of the charges that are interacting nor on their positions.

The numerical value of $k$ depends on what system of units we are using. Since charge is a new physical quantity, we could always choose the units of charge so that the constant $k$ comes out simple.
This is what is done in CGS units:

**CGS units:**
- Length - cm, centimeters
- Time - s, seconds
- Mass - g, grams
- Force - dyn, dyne
- Energy - erg, erg

conversion to MKS
- cm = 10^{-2} m
- s = s
- g = 10^{-3} kg
- dyn = 10^{-5} N
- erg = 10^{-7} J

In CGS units, one measures the unit of charge so that \( k = 1 \). The unit of charge in CGS is the "esu" or electrostatic unit. The esu is defined so that 2 charges, each with charge 1 esu, separated by a distance 1 cm, will feel a force of 1 dyne.

\[
1 \text{ dyne} = \frac{(1 \text{ esu})^2}{(1 \text{ cm})^2}
\]

\[\Rightarrow 1 \text{ esu} = \text{cm} \sqrt{\text{dyne}}\]

In MKS (also called SI, or standard) units, the charge was defined historically to be the "Coulomb" or coul. In MKS units

\[
k = 8.988 \times 10^9 \, \frac{\text{N} \cdot \text{m}^2}{\text{Coul}^2}
\]

Conversion of charge from MKS to CGS

\[
1 \text{ coul} = 2.998 \times 10^9 \text{ esu}
\]
proof: Consider two \( \pm \) coul charges separated by \( 1 \) m. The force between them has magnitude

\[
F = \frac{k (\text{Coul})^2}{(1 \text{ m})^2} = 8.988 \times 10^9 \text{ Nm}^2 \cdot \frac{\text{Coul}^2}{\text{Coul}^2} = 8.988 \times 10^9 \text{ N}
\]

Suppose \( \text{1 Coul} = \text{Q esu} \) then we have

\[
F = 8.988 \times 10^9 \text{ N} = 8.988 \times 10^9 \times 10^5 \text{ dynes} = \frac{(\text{Q esu})^2}{(10^2 \text{ cm})^2}
\]

\[
(\text{Q esu})^2 = 8.988 \times 10^{18} \text{ dyn cm}^2
\]

\[
\text{Q esu} = 2.998 \times 10^9 \text{ cm}\sqrt{\text{dyn}}
\]

Since \( \text{esu} = \text{cm}\sqrt{\text{dyn}} \) we have \( \text{Q} = 2.998 \times 10^9 \)

So \( \text{1 coul} = 2.998 \times 10^9 \text{ esu} \)

In this course we will be using CGS units.
In MKS units it is customary to write the constant $k$ as $k = \frac{1}{4\pi \varepsilon_0}$ so Coulomb's law becomes

$$F = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21}$$

$\varepsilon_0$ is called the "permittivity of free space".

The value of the magnitude of the electron charge is

$$e = 4.8023 \times 10^{-10} \text{ esu} = 1.6022 \times 10^{-19} \text{ coul}$$

Note Coulomb's law has exactly the same form as Newton's law of gravitation.

$$F = \frac{G m_1 m_2}{r_{21}^2} \hat{r}_{21}$$

$m_1$, mass of particle 1 at $r_1$

$m_2$, mass of particle 2 at $r_2$

$$G = 6.67 \times 10^{-11} \text{ m}^3/\text{kg s}^2$$

$$= 6.67 \times 10^{-11} \cdot \frac{(10^2 \text{cm})^3}{(10^3 \text{g}) \text{s}^2} = 6.67 \times 10^{-8} \text{ cm}^3/\text{g s}^2$$

Since $m_1 > 0$, $m_2 > 0$, the (-) sign gives always an attractive gravitational force.
Interestingly to compare the Coulomb and gravitational force between an electron and a proton:

- Electron mass: \( m_e = 9.11 \times 10^{-28} \text{ g} \)
- Proton mass: \( m_p = 1.67 \times 10^{-27} \text{ g} \)

\[
\frac{F_{\text{elec}}}{F_{\text{grav}}} = \frac{\frac{e^2}{r^2}}{\frac{G m_e m_p}{r^2}} = \frac{e^2}{G m_e m_p}
\]

Since both Coulomb and gravity \( \sim \frac{1}{r^2} \) the ratio is independent of the separation \( r \).

\[
\frac{F_{\text{elec}}}{F_{\text{grav}}} = \frac{(4.80 \times 10^{10} \text{ esu})^2}{(6.67 \times 10^{-8} \text{ cm}^3 / \text{s}^2)} \frac{(9.11 \times 10^{-28} \text{ g}) (1.67 \times 10^{-27} \text{ g})}{(9.11 \times 10^{-28} \text{ g}) (1.67 \times 10^{-27} \text{ g})}
\]

\[
= 2.27 \times 10^{39} \frac{\text{esu}^2}{(\text{cm}^3 \text{ g} / \text{s}^2)} = 2.27 \times 10^{39} \frac{\text{dyn}}{\text{g} \text{ cm} / \text{s}^2}
\]

To see that units are correct, use \( \text{esu}^2 = \text{cm}^2 \text{ dyn} \)

\[
= \text{cm}^2 \left( \frac{\text{g cm}}{\text{s}^2} \right) = \frac{\text{cm}^3 \text{ g}}{\text{s}^2}
\]

Electric force \( \approx 10^{39} \) times stronger than gravitational force. This is why we can ignore gravity when considering problems at the atomic scale.
Only when we are considering "large" objects (balls, planets) where \( m \) is large and the object is electrically neutral or close to it (equal number of electrons and protons) will \( F_{\text{grav}} \) dominate over \( F_{\text{elec}} \).

Coulomb's force is directed along the direction from \( q_1 \) to \( q_2 \).

This is reasonable since for stationary charges in uniform space the only direction that is uniquely defined is \( \overrightarrow{F_{\text{21}}} \).

There are simply no other vector quantities around, from which one could try to make the vector force \( \overrightarrow{F} \). This will no longer be so if one or both of the charges are moving. Then the velocities \( \overrightarrow{v_1} \) and \( \overrightarrow{v_2} \) give other vector directions; similarly, accelerations \( \overrightarrow{a_1} \) and \( \overrightarrow{a_2} \) need not be zero.
Principle of Superposition

When there are more than two charges, the total electric force on a given charge \( q_i \) is the sum of Coulomb forces from all other charges \( q_j \).

\[
\mathbf{F}_3 = \frac{g_1 q_2}{r_{31}^2} \hat{r}_{31} + \frac{g_2 q_3}{r_{32}^2} \hat{r}_{32}
\]

Similarly,

\[
\mathbf{F}_2 = \frac{g_1 q_2}{r_{21}^2} \hat{r}_{21} + \frac{g_3 q_2}{r_{23}^2} \hat{r}_{23}
\]

\[
\mathbf{F}_1 = \frac{g_2 q_1}{r_{12}^2} \hat{r}_{12} + \frac{g_3 q_1}{r_{13}^2} \hat{r}_{13}
\]

For \( N \) charges \( q_i \), \( i = 1, N \), force on a particular \( q_i \) is

\[
\mathbf{F}_i = \sum_{j \neq i} \frac{q_i q_j}{r_{ij}^2} \hat{r}_{ij} = q_i \sum_{j \neq i} \frac{q_j}{r_{ij}^2} \hat{r}_{ij}
\]