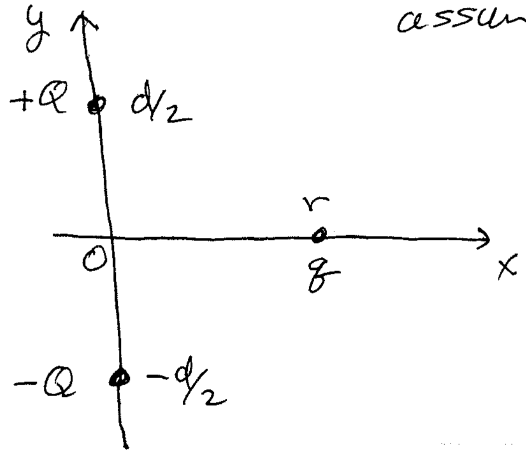
Example



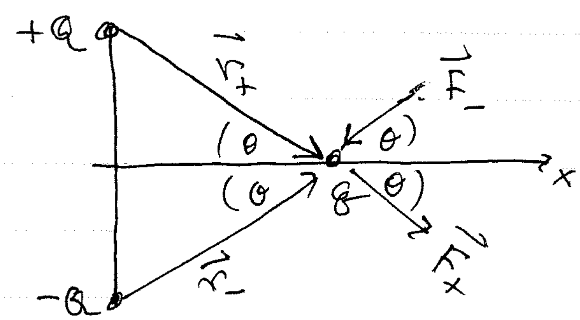
assume $q>0$

What is force on q due to $+Q$ and $-Q$?

q is at $r\hat{x}$
$+Q$ is at $\frac{d}{2}\hat{y}$
$-Q$ is at $-\frac{d}{2}\hat{y}$

We can do this problem geometrically, or we can do it algebraically

Geometric metod

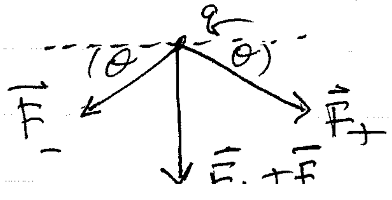


the distance between q and $+Q$ is
$$r_+ = \sqrt{r^2+(d/2)^2}$$
the distance between q and $-Q$ is
$$r_- = \sqrt{r^2+(d/2)^2} = r_+$$

Since $r_+ = r_-$, the force $\vec{F}_+$ on q from $+Q$ has same magnitude as the force $\vec{F}_-$ on q from $-Q$

$$F_+ = F_- = \frac{qQ}{r_+^2} = \frac{qQ}{r^2+(\frac{d}{2})^2}$$

But the directions of $\vec{F}_+$ and $\vec{F}_-$ ~~are~~ not the same.
$\vec{F}_+$ points away from $+Q$, $\vec{F}_-$ points into $-Q$.
when we add $\vec{F}_+ + \vec{F}_-$ we see it points in $-\hat{y}$ direction



The x-components of the forces are

$$F_{+x} = |\vec{F}_+| \cos\theta$$

$$F_{-x} = -|\vec{F}_-| \cos\theta$$

since $|\vec{F}_+| = |\vec{F}_-|$ we have $F_{+x} = -F_{-x}$
and when we add them they cancel

The y-components of the forces are

$$F_{+y} = -|\vec{F}_+| \sin\theta$$

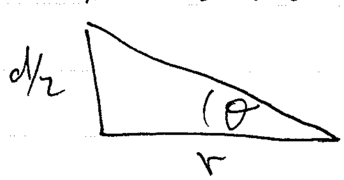
$$F_{-y} = -|\vec{F}_-| \sin\theta$$

these are equal. When we add them we get

$$F_y = F_{+y} + F_{-y} = -2|\vec{F}_+| \sin\theta$$

Hence $\vec{F} = -2 \dfrac{qQ}{r^2 + \left(\frac{d}{2}\right)^2} \sin\theta \, \hat{y}$

But $\sin\theta = \dfrac{d/2}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2}}$



So $\boxed{\vec{F} = \dfrac{-2qQ\,(d/2)}{\left(r^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{y}}$

Algebraic

Let $\vec{r}_q = r\hat{x}$ position of q

$\vec{r}_Q = \frac{d}{2}\hat{y}$ position of $+Q$

$\vec{r}_{-Q} = -\frac{d}{2}\hat{y}$ position of $-Q$

$$\vec{r}_+ \equiv \vec{r}_q - \vec{r}_Q = r\hat{x} - \frac{d}{2}\hat{y}$$

$$|\vec{r}_+| = \sqrt{r_{+x}^2 + r_{+y}^2 + r_{+z}^2} = \sqrt{r^2 + \left(\frac{d}{2}\right)^2}$$

$$\hat{r}_+ = \frac{r\hat{x} - \frac{d}{2}\hat{y}}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2}} = \frac{\vec{r}_+}{|\vec{r}_+|}$$

Similarly

$$\vec{r}_- = \vec{r}_q - \vec{r}_{-Q} = r\hat{x} + \frac{d}{2}\hat{y}$$

$$|\vec{r}_-| = \sqrt{r_{-x}^2 + r_{-y}^2 + r_{-z}^2} = \sqrt{r^2 + \left(\frac{d}{2}\right)^2}$$

$$\hat{r}_- = \frac{r\hat{x} + \frac{d}{2}\hat{y}}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2}} = \frac{\vec{r}_-}{|\vec{r}_-|}$$

$$\vec{F} = \vec{F}_+ + \vec{F}_- = \frac{qQ}{r_+^2}\hat{r}_+ + \frac{q(-Q)}{r_-^2}\hat{r}_-$$

$$= \frac{qQ}{r^2 + \left(\frac{d}{2}\right)^2} \cdot \frac{\left(r\hat{x} - \frac{d}{2}\hat{y}\right)}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2}} + \frac{(-qQ)}{r^2 + \left(\frac{d}{2}\right)^2} \cdot \frac{\left(r\hat{x} + \frac{d}{2}\hat{y}\right)}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2}}$$

$$= \frac{qQ}{\left[r^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}} \left\{\left(r\hat{x} - \frac{d}{2}\hat{y}\right) - \left(r\hat{x} + \frac{d}{2}\hat{y}\right)\right\}$$

$$\boxed{\vec{F} = \frac{-qQd\,\hat{y}}{\left[r^2 + (d/2)^2\right]^{3/2}}}$$

same answer as by geometric method.

$$\vec{F} = \frac{-qQd}{\left[r^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}} \hat{y}$$

true for all r and d.

Consider two limits: $r \gg d$ and $r \ll d$.

① $r \gg d$ we want to approx the term

$$\frac{1}{\left[r^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}} = \frac{1}{r^3\left[1 + \left(\frac{d}{2r}\right)^2\right]^{3/2}}$$

when $r \gg d$ then $\frac{d}{2r} \ll 1$ is <u>very</u> small
to lowest order approximation we can ignore
$\left(\frac{d}{2r}\right)^2$ compared to 1 and thus get

$$\vec{F} \approx -\frac{qQd}{r^3} \hat{y} \qquad \text{decreases as } \frac{1}{r^3} \text{ rather than } \frac{1}{r^2}$$

this is the behavior of an electric <u>dipole</u>

$\vec{F}$ decreases faster than $\frac{1}{r^2}$ since as r gets further away, the force from $+Q$ and $-Q$ come closer to cancelling. ~~Very far away $r \ggg d$, the $+Q$ and $-Q$ look like~~

② $r \ll d$ Now we want to approx the term

$$\frac{1}{\left[r^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}} = \frac{1}{\left(\frac{d}{2}\right)^3\left[\left(\frac{2r}{d}\right)^2 + 1\right]^{3/2}}$$

when $r \ll d$ we now ignore the $\frac{2r}{d}$ term

But then we get just $\frac{1}{(\frac{d}{2})^3}$ and

$$\vec{F} \simeq \frac{-qQd}{(\frac{d}{2})^3}\,\hat{y} = \frac{-8qQ}{d^2}\,\hat{y}$$

But this has lost <u>all</u> the r-dependence - it is just the force when q is at r=0, exactly midway between +Q and -Q.

To do better we want to make approx to lowest non zero order in $\frac{2r}{d}$

Consider $\quad f(\epsilon) = \frac{1}{(1+\epsilon)^{3/2}} \quad$ where $\epsilon = \left(\frac{2r}{d}\right)^2$ is small

We can use a <u>Taylor series</u> approx to write

$$f(\epsilon) \approx f(0) + f'(0)\,\epsilon + \tfrac{1}{2} f''(0)\,\epsilon^2 + \cdots$$

$$f(0) = 1, \quad f'(\epsilon) = -\frac{3}{2}\frac{1}{(1+\epsilon)^{5/2}} \Rightarrow f'(0) = -3/2$$

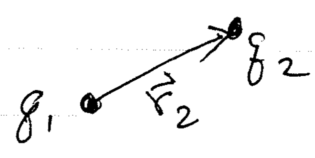
So $\quad \frac{1}{(1+\epsilon)^{3/2}} \simeq 1 - \frac{3}{2}\epsilon$

So $\quad \frac{1}{\left[1+\left(\frac{2r}{d}\right)^2\right]^{3/2}} \simeq 1 - \frac{3}{2}\left(\frac{2r}{d}\right)^2 = 1 - 6\frac{r^2}{d^2}$

$$\vec{F} = \frac{-qQd}{(\frac{d}{2})^3}\left[1 - 6\frac{r^2}{d^2}\right]\hat{y} = -\frac{8qQ}{d^2}\left[1 - 6\frac{r^2}{d^2}\right]\hat{y}$$

Energy of a charge configuration

Consider two charges q_1 and q_2. For convenience, choose the origin of the coordinates so q_1 is at $\vec{r}_1 = 0$ and q_2 is at $\vec{r}_2$



What was the work done <u>on</u> the charges to bring them into this configuration?

We ignore whatever work may have been done to create the charges q_1 and q_2 in and of themselves, and just want to know the work it took to move them into these positions, starting from an initial state in which they are infinitely far apart from each other (and so not exerting any force on each other).

We start with q_1 and q_2 both at infinity. Leaving q_2 at infinity we can now move q_1 in to the origin. This takes <u>no</u> work since q_1 and q_2 always stay infinitely apart from each other and so exert no force on each other.

Next we want to compute the work done on q_2 to move it in from infinity to the position $\vec{r}_2$, acting against the Coulomb force $\vec{F}_2$ on q_2 from q_1.

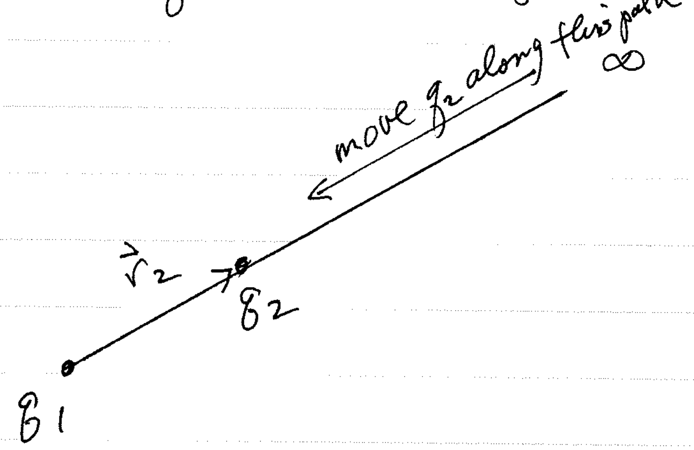
The force we have to exert on q_2 (we will call this the "mechanical" force – ie the force of our pushing q_2 with some mechanical means) $\vec{F}_{mech}$ should be exactly equal in magnitude but opposite in direction to the Coulomb force $\vec{F}_2$ that acts on q_2 due to the presence of q_1.

$$\vec{F}_{mech} = -\vec{F}_2$$

In this way the total force on q_2 will be $\vec{F}_{mech} + \vec{F}_2 = 0$ and we can gently move q_2 into position without accelerating it and giving it a velocity. If we use $\vec{F}_{mech} = -\vec{F}_2$, q_2 will wind up at $\vec{r}_2$ at rest, which is what we want for electrostatics.

$$\text{So} \quad \vec{F}_{mech} = -\vec{F}_2 = -\frac{q_1 q_2}{r_2}\hat{r}_2$$

Imagine now that we move q_2 in from infinity along a path that is radial with respect to q_1 at the origin



The work done on q_2 is just the integral of force x distance. Since $\vec{F}_{mech}$ points in the same direction as the particles motion we can write

$$W = \int_{\infty}^{r_2} F_{mech}(r)\, dr$$

with $F_{mech}(r) = \frac{q_1 q_2}{r^2}$ the force on q_2 when it is a distance r from the origin

$$= -\int_{\infty}^{r_2} \frac{q_1 q_2}{r^2}\, dr$$

$$= \left[\frac{q_1 q_2}{r}\right]_{\infty}^{r_2} = \frac{q_1 q_2}{r_2} - 0$$

$$W = \frac{q_1 q_2}{r_2}$$

where $r_2 = |\vec{r}_2|$ is the distance between q_1 and q_2

More generally, if q_1 is at $\vec{r}_1$ and q_2 is at $\vec{r}_2$, then

$$W = \frac{q_1 q_2}{r_{21}}$$

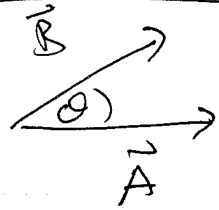
where $r_{21} = |\vec{r}_{21}| = |\vec{r}_2 - \vec{r}_1|$ is again the distance between q_1 and q_2

Next we want to show that we get the same result for W for <u>any</u> path that takes q_2 from ∞ to $\vec{r}_2$ – it does <u>not</u> need to be a radial path, any path will do!

In general, when the force is <u>not parallel</u> to the direction of motion along the path, the work along any <u>small</u> segment of the path $d\vec{s}$ is the dot product $\vec{F} \cdot d\vec{s}$. Adding up all work along all such small segments gives the work done on the particle as the <u>line integral</u>

Quick review of dot product (or inner product)

geometrically



for two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$

algebraically

for
$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$
$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

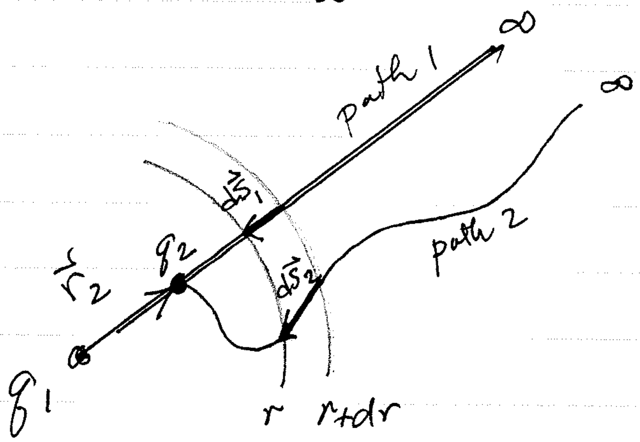
A_x, A_y, A_z are the "components" of $\vec{A}$ in directions $\hat{x}, \hat{y}, \hat{z}$ respectively

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$W = \int_{\vec{r}_{start}}^{\vec{r}_{end}} \vec{F} \cdot d\vec{s}$$

For our electrostatic problem we want to compute

$$W = \int_{\infty}^{\vec{r}_2} \vec{F}_{mech} \cdot d\vec{s}$$ along any path starting at ∞ and ending at $\vec{r}_2$.



Call the radial path we already considered "path 1". Consider any other path call it "path 2"

Consider the work done on each path in moving q_2 from a distance $r+dr$ to the closer distance r. Assume dr is small enough that $\vec{F}_{mech} = -\vec{F}_2$ is approximately constant over this distance, and the the segment of the path going from $r+dr$ to r is approximately a constant vector.

$d\vec{s}_1$ is the segment of path from $r+dr$ to r along the radial path 1

$d\vec{s}_2$ is the segment of path from $r+dr$ to r along the arbitrary path 2.

Now let $\hat{r}$ be the outward radial direction from q_1 at the origin. Then

$$\vec{F}_{mech} = -\frac{q_1 q_2}{r^2}\hat{r} = -\vec{F}_2$$

is the force we exert on q_2 when it is a distance r from q_1.

The work done in moving q_2 along path 1 from $r+dr$ to r is then

$$dW_1 = \vec{F}_{mech} \cdot d\vec{s}_1$$

Here $d\vec{s}_1 = -dr\,\hat{r}$ (−) sign since $d\vec{s}$ points inward

So $$dW_1 = \left(-\frac{q_1 q_2}{r^2}\hat{r}\right)\cdot\left(-dr\,\hat{r}\right)$$

$$= \frac{q_1 q_2}{r^2}dr \qquad (\text{since } \hat{r}\cdot\hat{r} = 1)$$

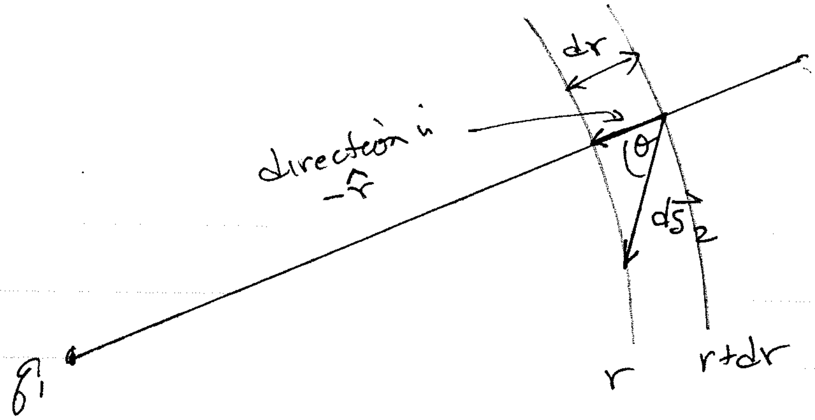
Now consider the work done along path 2

$$dW_2 = \vec{F}_{mech}\cdot d\vec{s}_2$$

$$= \left(-\frac{q_1 q_2}{r^2}\hat{r}\right)\cdot d\vec{s}_2$$

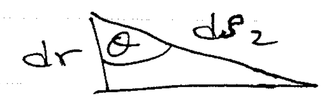
$$= \frac{q_1 q_2}{r^2}\,|d\vec{s}_2|\cos\theta$$

where θ is the angle shown below and $|d\vec{s}_2| \equiv ds_2$ is the magnitude of $d\vec{s}_2$



From trigonometry we can relate the length ds_2 to dr

$$ds_2 = \frac{dr}{\cos\theta}$$



$$\cos\theta = \frac{dr}{ds_2}$$

So

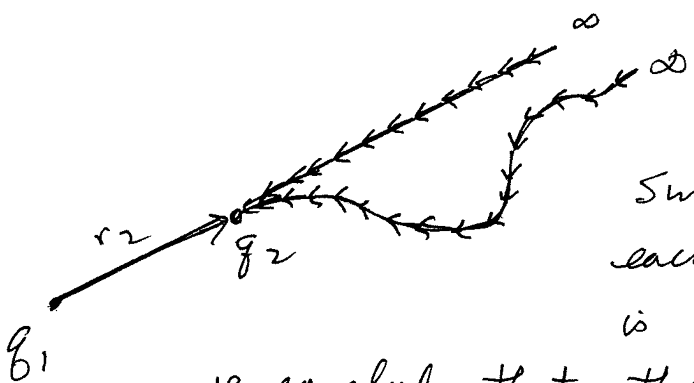
$$dW_2 = \frac{q_1 q_2}{r^2} \frac{dr}{\cos\theta} \cos\theta$$

$$= \frac{q_1 q_2 \, dr}{r^2} = dW_1$$

So $dW_2 = dW_1$

Thus the work to move q_2 from a distance $r+dr$ to r is independent of the path taken.

We can divide the total work done on any path as the sum of works done on each small segment from $r+dr$ to r.



Since the work done on each segment from $r+dr$ to r is independent of the path we conclude that the total work along the entire path is independent of the particular path taken

$$W_1 = W_2$$

So the work done to move q_1 and q_2 into positions ~~from~~ at $\vec{r}_1$ and $\vec{r}_2$ respectively is

$$\boxed{W = \frac{q_1 q_2}{r_{21}}}$$

independent of paths taken — also independent of which charge is moved into position first.

~~The~~ Similarly we can consider the work done to move 3 charges q_1, q_2, q_3 into positions $\vec{r}_1, \vec{r}_2, \vec{r}_3$.

Start with all 3 charges at infinity.

1) move q_1 to $\vec{r}_1$ — this costs no work $W_1 = 0$
2) move q_2 to $\vec{r}_2$ — this costs $W_2 = \dfrac{q_1 q_2}{r_{21}}$
3) now move q_3 in from infinity to $\vec{r}_3$. the work done in this last step is

$$W_3 = \int_\infty^{\vec{r}_3} \vec{F}_{mech} \cdot d\vec{s} \qquad \text{where } \vec{F}_{mech} = -\vec{F}_3$$

$\vec{F}_3$ is Coulomb force on q_3 due to both q_1 and q_2.

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32} = \frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + \frac{q_2 q_3}{r_{32}^2}\hat{r}_{32}$$

force on q_3 due to q_1 — force on q_3 due to q_2

$$W_3 = \int_\infty^{\vec{r}_3} \vec{F}_{mech} \cdot d\vec{s} = -\int_\infty^{\vec{r}_3} \vec{F}_3 \cdot d\vec{s}$$

$$= -\int_\infty^{\vec{r}_3} \vec{F}_{31} \cdot d\vec{s} - \int_\infty^{\vec{r}_3} \vec{F}_{32} \cdot d\vec{s}$$

independent of q_2 — independent of q_1

$$= \frac{q_1 q_3}{r_{31}} + \frac{q_3 q_2}{r_{32}}$$

Total work to move all three charges into position is

$$W = W_1 + W_2 + W_3$$

$$= 0 + \frac{q_1 q_2}{r_{21}} + \frac{q_1 q_3}{r_{31}} + \frac{q_3 q_2}{r_{32}}$$

one term for each pair of charges

Similarly if we had N charges $q_1, q_2, \dots, q_N$ at positions $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$

$$W = \sum_{\substack{\text{pairs} \\ (i,j)}} \frac{q_i q_j}{r_{ji}}$$

one term in sum for each distinct pair of charges (i,j)

or we could write

$$W = \underbrace{\frac{1}{2}}_{\text{factor of } \frac{1}{2} \text{ since double sum counts all pairs twice}} \underbrace{\sum_{i=1}^{N}}_{\text{sum over all charges } i} \underbrace{\sum_{j=1}^{N}}_{\text{sum over all charges } j} \frac{q_i q_j}{r_{ji}}$$

in double sum we must exclude the terms where $i=j$

Since the work done depends only on the final configuration of the charges $\{\vec{r}_i\}$ and not on the order in which they were assembled or the paths along which they were moved into their final configuration, it is useful to think of this work done to assemble the configuration as a potential energy stored in the configuration

$$U = \frac{1}{2} \sum_{i} \sum_{\substack{j \\ i \neq j}} \frac{q_i q_j}{r_{ij}}$$

Since work done does not depend on the path we say that the electrostatic force is conservative

is the electrostatic potential energy of the electrostatic charge configuration.

As with all potential energies U is arbitrary within an additive constant. We choose this constant so $U=0$ when all charges are infinitely far away from each other.

The charges $\{q_i\}$ at positions $\{\vec{r}_i\}$ in general will be in a static state only when there are other forces (for example mechanical forces) that counterbalance the electrostatic Coulomb forces

If $U > 0$, then is we suddenly released these of the mechanical forces, the charges would fly apart from each other and move off infinitely far away from each other. After a long time one would find $U_{final} = 0$ but the sum of kinetic energies of all the charges would be equal to the initial U.

This is an example of energy conservation.

For a pair q_1 and q_2

$$U = \frac{q_1 q_2}{r_{21}}$$

$U > 0$ if q_1 and q_2 are same sign
since charges repel we had to do work to move them together

$U < 0$ if q_1 and q_2 are opposite sign
since charges attract, they will release energy as the charges move closer together. Now we would have to do work to separate the charges a greater distance.