The Electric Field

Consider charges $\{q_i\}$ at positions $\{\vec{r}_i\}$ $\quad i=1,\dots,N$

If we now put a charge q_0 at position $\vec{r}_0$, the force it feels from the other charges is

$$\vec{F}_0 = \sum_{i=1}^{N} \frac{q_0 q_i}{r_{0i}^2}\,\hat{r}_{0i} = q_0 \sum_{i=1}^{N} \frac{q_i}{r_{0i}^2}\,\hat{r}_{0i}$$

where $\vec{r}_{0i} = \vec{r}_0 - \vec{r}_i$

$\vec{F}_0$ is proportional to q_0 so we define

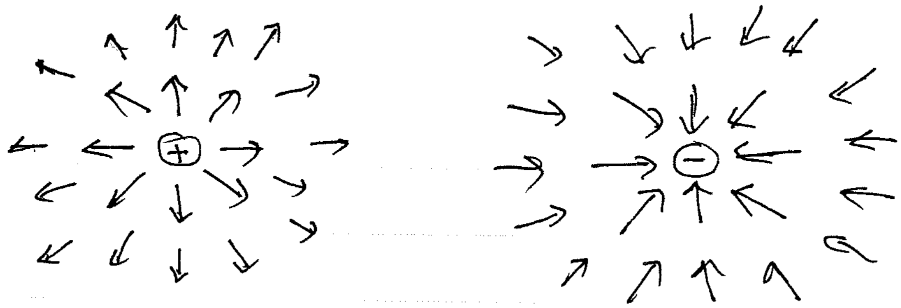
$$\frac{\vec{F}_0}{q_0} \equiv \vec{E} = \sum_{i=1}^{N} \frac{q_i}{r_{0i}^2}\,\hat{r}_{0i}$$ <u>electric field</u> at $\vec{r}_0$

The electric field $\vec{E}$ from a static configuration of charges is a <u>vector</u> <u>function</u> of space such that $\vec{E}(\vec{r})$ gives the force per unit charge that a charged particle would feel if it was placed at position $\vec{r}$.

Units of $\vec{E}$ are dyn/esu in CGS units

are N/coul in MKS units

To visualize the electric field, one method is to pick points in space and draw the $\vec{E}$ field vector at ~~that~~ those points.



$\vec{E}$ from $+q$

$+q$ is a source

$\vec{E}$ field points outward from $+q$ and diverges at $+q$

$\vec{E}$ from $-q$,

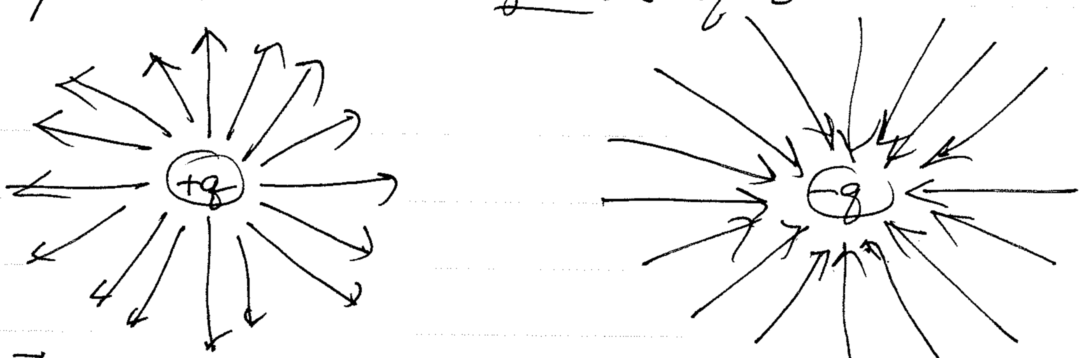
$-q$ is a sink

$\vec{E}$ field points into $-q$ and diverges at $-q$

direction of $\vec{E}$ is not defined exactly at q.

Field lines

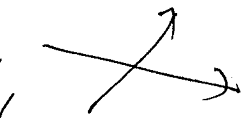
Another popular visualization is "field lines". Field lines are continuous directed lines. The tangent to the ~~are~~ lines lie in the direction of $\vec{E}$. The density of lines is proportional to the magnitude of $\vec{E}$

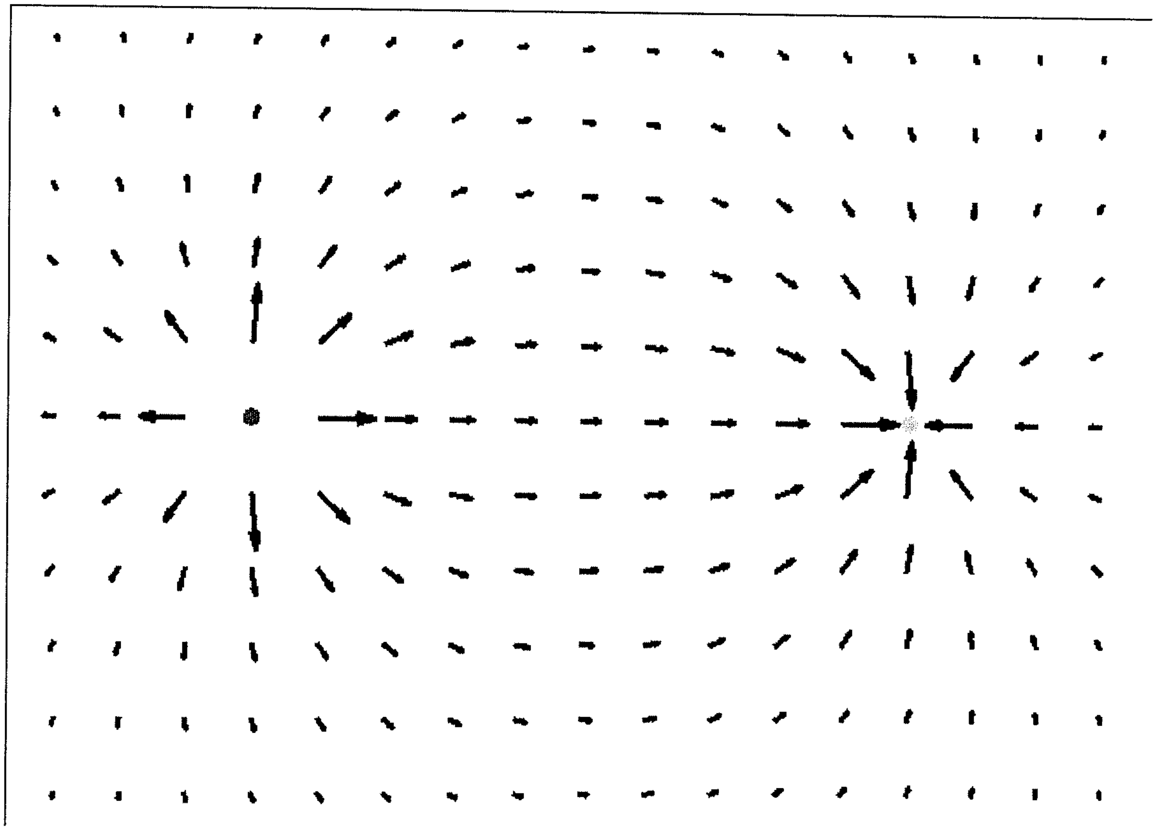


$\vec{E}$ radially outward

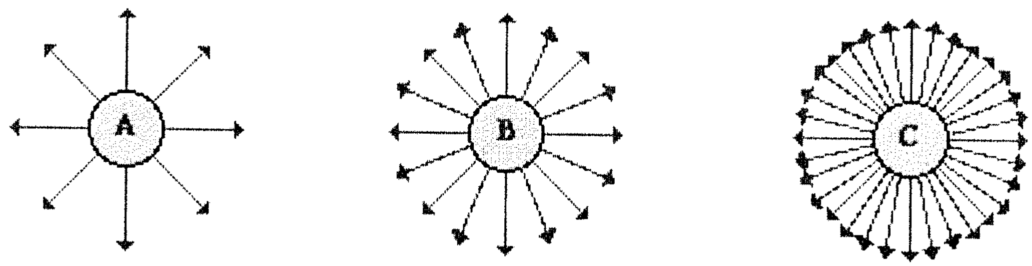
magnitude increases as approach q

Electric field lines are continuous and only can begin or end at a place where there is charge.

Field lines cannot cross ie can't be like  since at the point of crossing there would not be a unique direction for the force

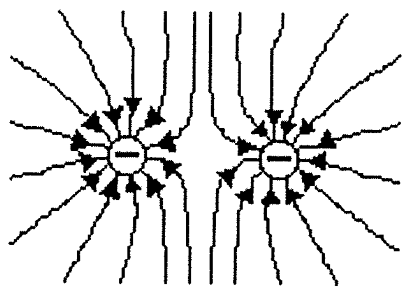


Density of Lines in Patterns

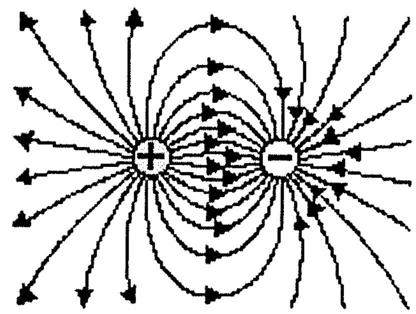


The density of electric field lines around these three objects reveals that the quantity of charge on C is greater than that on B which is greater than that on B.

Other Charge Configurations

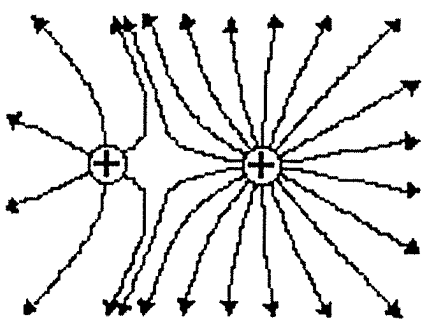


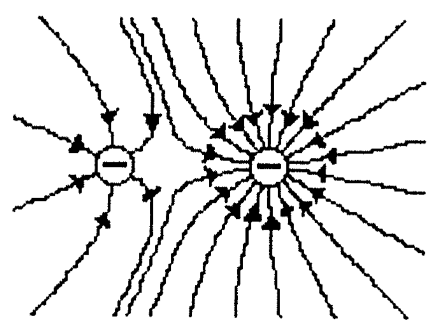
Two Negatively Charged Objects

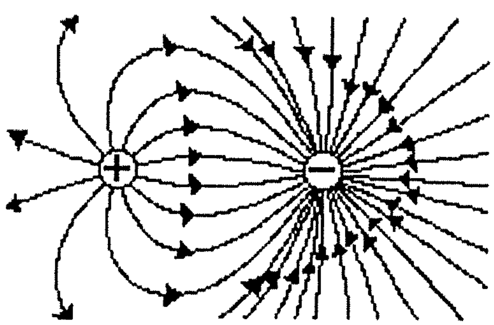


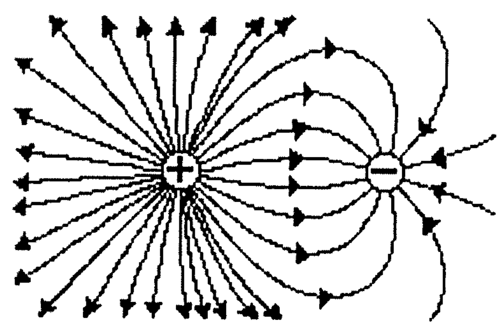
A Positively and a Negatively Charged Object

Electric Field Line Patterns for Objects with Unequal Amounts of Charge

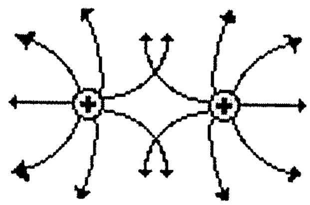








Field Lines CANNOT cross each other



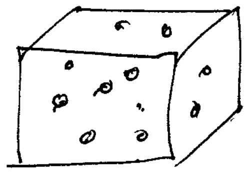
Charge distributions

So far we have considered charges as point-like objects at discrete positions in space. This is not a bad description at the atomic level where the electron and proton are indeed very close to point-like objects (at least on the scale of 10^{-15} m). And since all charge distributions are in principle just collections of protons and electrons (and antiprotons and positrons and ....) it is in principle also correct at ~~and~~ any length scale.

However in classical E&M when we are dealing with macroscopically large objects in where there are ~~huge~~ huge numbers of point-like charges, it becomes convenient to ~~def~~ define a smooth continuous charge density function

$$\rho(\vec{r}) = \rho(x,y,z) \qquad \text{charge density}$$

This is very similar, for example, to hydrodynamic discriptions of fluids — there one defines a smooth continuous density function $M(x,y,z)$ even though at the atomic scale the fluid is comprised of discrete atoms or molecules.



$q_{tot} = \sum_{i \in box} q_i$ total charge in box

imaginary box located at position $\vec{r}$

↑ volume ΔV large on atomic scale but very small on macroscopic scale

One defines the charge density ρ at position $\vec{r}$ by

$$\rho(\vec{r}) = \frac{q_{tot}}{\Delta V}$$

ρ has units of charge/volume $=$ esu/cm^3 in CGS

For discrete point charges we had

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|^2} \widehat{\vec{r}-\vec{r}_i}$$

$\widehat{\vec{r}-\vec{r}_i} = \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|}$ is unit vector from $\vec{r}_i$ to $\vec{r}$

Then for a continuous charge density $\rho(\vec{r})$ we replace the sum over q_i with an integral over space

Since the contribution to the field $E(\vec{r})$ due to the charge $\rho(\vec{r}\,')\Delta V$ at $\vec{r}\,'$ is $\frac{\rho(\vec{r}\,')\Delta V}{|\vec{r}-\vec{r}\,'|^2} \widehat{\vec{r}-\vec{r}\,'}$

The total electric field is

$$\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \widehat{\vec{r}-\vec{r}\,'} \, dV'$$

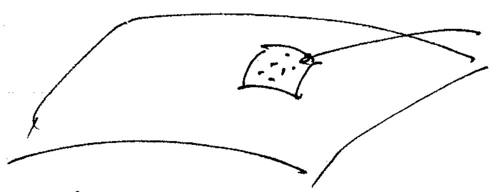
↑ infinitesimal volume at position $\vec{r}\,'$

or in terms of coordinates x, y, z

$$\vec{E}(x,y,z) = \int \frac{\rho(x',y',z')}{|\vec{r}-\vec{r}'|^2} \, \widehat{\vec{r}-\vec{r}'} \, dx'dy'dz'$$

<u>Surface charge</u>

In some physical problems we will encounter the situation where the charge is confined to sit on an infinitesimally thin surface, but is otherwise smoothly distributed on the surface. This is called a <u>surface charge density σ</u>



Surface S

small area ΔA on surface contain charge
area ΔA is located at a position $\vec{r}$ on surface

$$\sigma(\vec{r}) = \frac{q_{tot}}{\Delta A}$$

q_{tot} is total charge in ΔA

surface charge density σ has units of charge/area or esu/cm^2 in CGS

<u>line charge</u>

Sometimes the charge is confined to sit on a one-dimensional line. In this case one can define a continuous <u>line charge density λ</u>

line

Δl

small length Δl at position $\vec{r}$

$$\lambda(\vec{r}) = \frac{q_{tot}}{\Delta l}$$

q_{tot} is total charge in Δl

line charge has units of charge/length or esu/cm in CGS

We can now compute $\vec{E}(\vec{r})$ for some simple charge distributions using the integral form of Coulomb's law

$$E(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \widehat{\vec{r}-\vec{r}\,'} \underbrace{dV'}_{}$$

infinitesimal volume such as $dx'dy'dz'$

Note the unit vector $\widehat{\vec{r}-\vec{r}\,'} \equiv \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$ points from the charge at $\vec{r}\,'$ to the point $\vec{r}$ where we are computing the electric field. As we integrate over the region containing the charge, the direction $\widehat{\vec{r}-\vec{r}\,'}$ continuously changes direction. This can make the computation of the integral very difficult for a general case. But in some situations where the charge density is very symmetric, the integral is relatively easy to do.

Note: for a surface charge density σ, the above integral is

$$\vec{E}(\vec{r}) = \int_S \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \widehat{\vec{r}-\vec{r}\,'} \, da'$$

integrate over surface containing charge

infinitesimal area on surface

for a line charge λ, the integral is

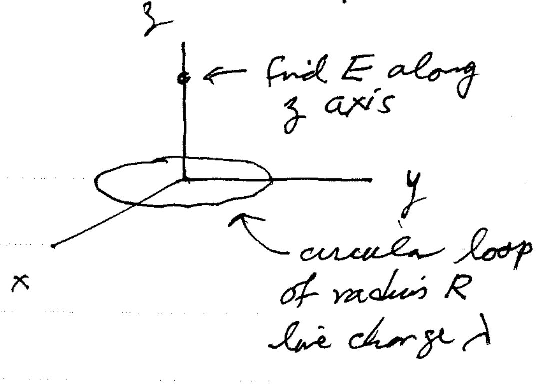
$$\vec{E}(\vec{r}) = \int_C \frac{\lambda(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \widehat{\vec{r}-\vec{r}\,'} \, dl'$$

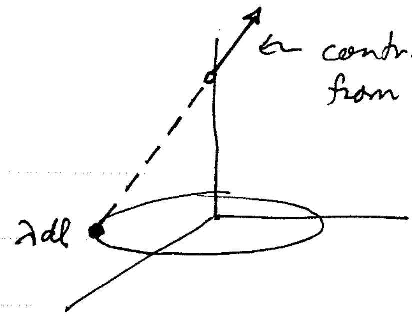
infinitesimal line segment

curve along which there is charge

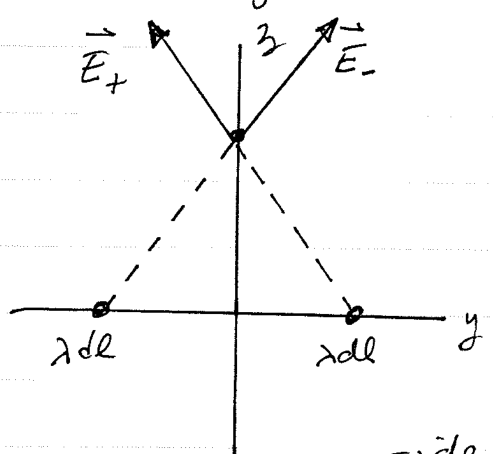
Example

Find $\vec{E}(\vec{r})$ along the axis going through a circular wire loop with constant line charge λ





As one integrates around the loop, the contribution to $\vec{E}$ from each segment of the loop changes direction. But integration becomes easy if we pair up points on opposite sides of the loop

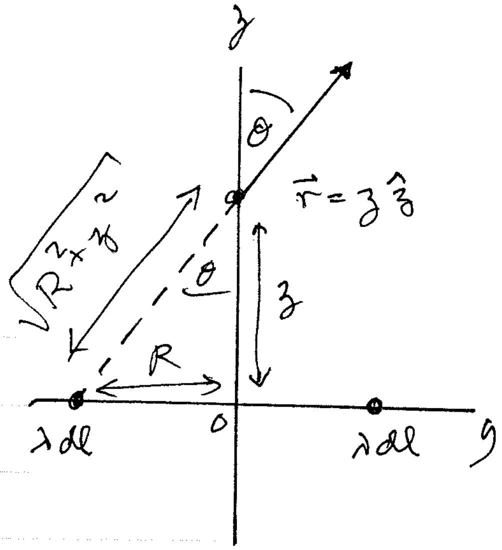


when we add $\vec{E}_-$ (the contribution to $\vec{E}$ from λdl at $y=-R$) to $\vec{E}_+$ (the contribution to $\vec{E}$ from λdl at $y=+R$) the result $\vec{E}_+ + \vec{E}_-$ lies along the z axis!

components of $\vec{E}_+$ and $\vec{E}_-$ along $\hat{z}$ are equal. Component of $\vec{E}_+$ and $\vec{E}_-$ in xy plane are equal in magnitude but opposite in direction

As we integrate around the loop the total $\vec{E}$ will therefore point in the $\hat{z}$ direction

Thus we only need to compute the z component of E from each segment λdl of the loop



the <u>magnitude</u> of the contribution to $\vec{E}$ from the line charge segment dl is

$$E = \frac{\lambda dl}{R^2 + z^2}$$

The z-component of this is

$$E\cos\theta = \left(\frac{\lambda dl}{R^2+z^2}\right)\frac{z}{\sqrt{R^2+z^2}}$$

$$= \frac{z\,\lambda\,dl}{(R^2+z^2)^{3/2}}$$

We now need to sum this up for all segments of the loop

$$E_z(z) = \int_{\text{loop}} dl\,\frac{z\lambda}{(R^2+z^2)^{3/2}}$$

but since integrand is a constant for all points on the loop, $\int dl$ = circumference of loop $2\pi R$

So for a point $\vec{r} = z\hat{z}$ on z axis

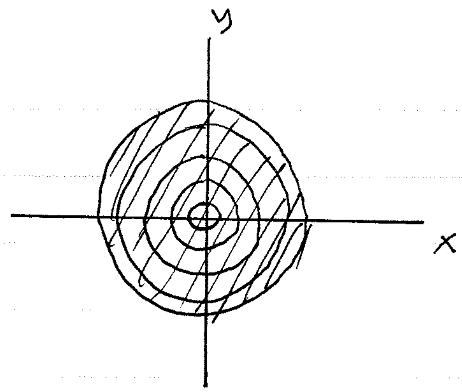
$$\boxed{\vec{E}(z) = \frac{2\pi R\, z\,\lambda}{(R^2+z^2)^{3/2}}\,\hat{z}}$$

$\vec{E}$ points in $\hat{z}$ direction

$2\pi R\lambda = Q$ total charge

Note: the above symmetry argument that made it easy to do the integral only works for $\vec{r}$ on the z-axis. We could not do it this way for a general position $\vec{r}$ off the z axis.

What is $\vec{E}$ along z axis for a disk of radius R with uniform surface charge density σ?



regard disk as a superposition of rings of small thickness dr

the ring at radius r has a line charge density

$$\lambda(r) = \sigma\, dr$$

to get $\vec{E}(z)$ from the disk, we just take $\vec{E}(z)$ from the ring and integrate over all rings from r = 0 to r = R

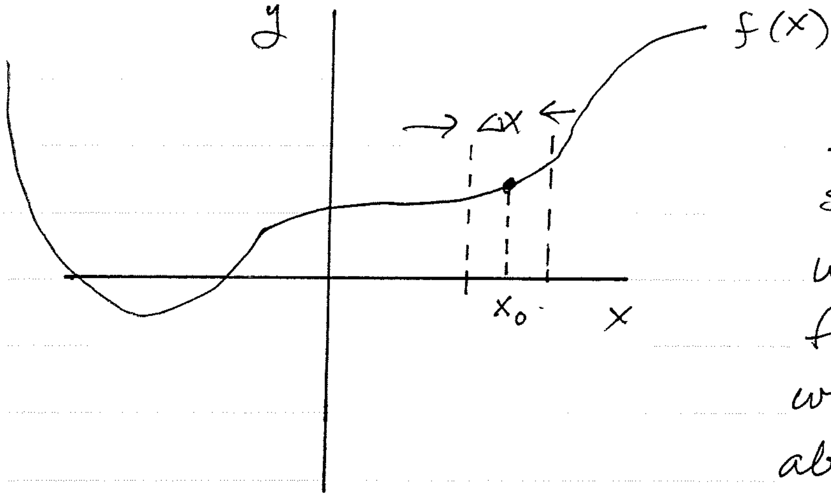
$$\vec{E}(z) = \int_0^R \frac{2\pi r z \overbrace{\sigma\, dr}^{\lambda(r)}}{(r^2+z^2)^{3/2}}\,\hat{z}$$

$$= 2\pi\sigma z \hat{z} \int_0^R \frac{r\, dr}{(r^2+z^2)^{3/2}}$$

$$= 2\pi\sigma z \hat{z} \left[\frac{-1}{(r^2+z^2)^{1/2}}\right]_0^R$$

$$\boxed{\vec{E}(z) = 2\pi\sigma z \hat{z}\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right]}$$

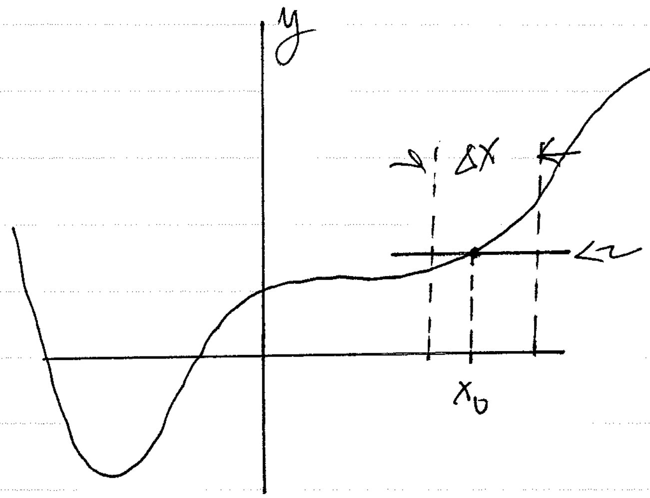
A physicist's Review of Taylor series



If $f(x)$ is a smooth single valued function of x, we want a simple approximation for $f(x)$ that will be good within a small window Δx about x_0.

Zeroth Approx: just replace $f(x)$ by its value at x_0

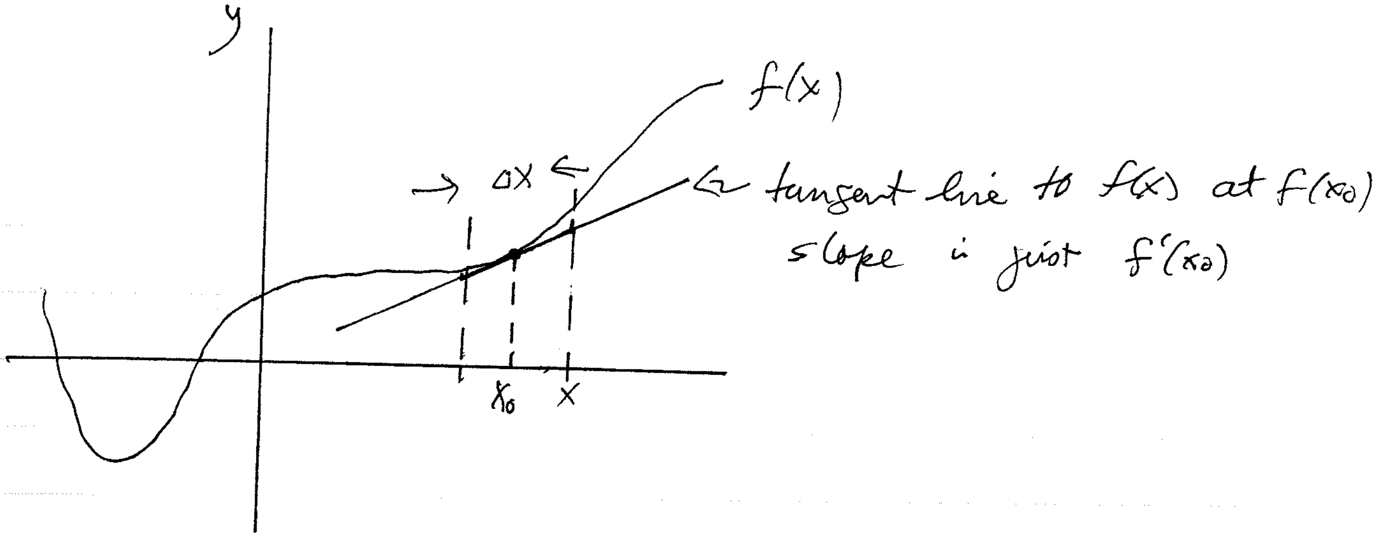
$$f(x) \approx f(x_0)$$



approx $f(x) \approx f(x_0)$

this approx replaces $f(x)$ by a constant

1st order Approx: We now want to make a better approx – something that will tell us if (and by how much) $f(x)$ changes if x changes from x_0.

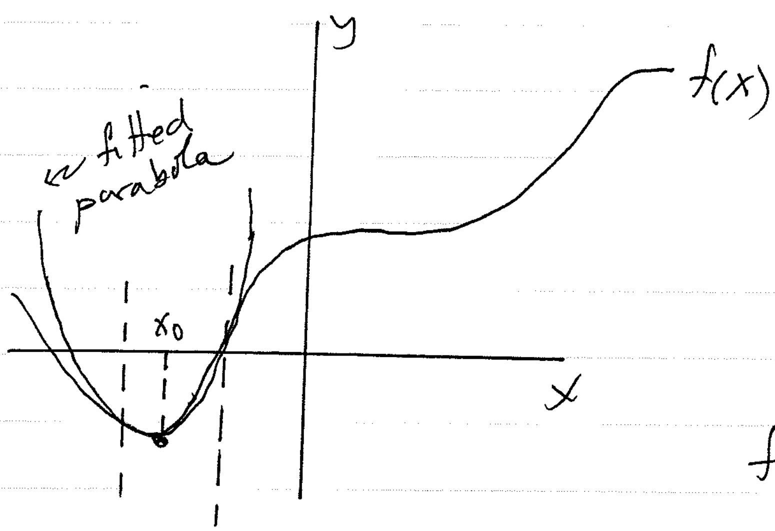


the 1st order approx just replaces $f(x)$ by its tangent line at x_0. The equation for this line is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

~~But so~~ 2nd order approx

But suppose $f'(x_0) = 0$ at x_0. Then 1st order approx is no better than zeroth order approx. We could improve things by fitting a parabola at x_0



The parabola that fits best is determined by the curvature of $f(x)$ at x_0. Curvature is $f''(x_0)$ and equ of parabola is

$$f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

Adding this 2nd order term improves the approx even when $f'(x_0) \neq 0$. So the 2nd order approx is thus

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

<u>higher order</u>

One should now imagine that one could improve the approximation by adding addition terms that are power law $(x-x_0)^n$. Without giving any derivation, this results in the <u>Taylor Series</u> for the function $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} f^{(n)} \frac{(x_0)(x-x_0)^n}{n!}$$

where $f^{(n)}(x_0)$ is the nth derivative of $f(x)$ evaluated at x_0. $f^{(0)}(x) \equiv f(x)$.

$n! = n(n-1)(n-2)(n-3) \cdots (1)$ is "n factorial"

with $0! \equiv 1$

Summing up $n = 0, 1, 2$ terms gives the 2nd order approx above. If one can sum up <u>all</u> the infinite number of terms, the Taylor series becomes exactly equal to $f(x)$ over some interval Δx about x_0. How wide is Δx depends on the specific analytical properties of the function $f(x)$.

In using the Taylor series in physics, we seldom will go beyond the 2nd order approx. Often we will stop at the 1st order.

We can generalize the Taylor series to a function of more than one variable in a straightforward way. For example, for a function of 3 spatial coordinates $f(x, y, z)$ we can write to 1st order

$$f(x,y,z) \approx f(x_0, y_0, z_0) + \frac{df(x_0,y_0,z_0)}{dx}(x-x_0)$$
$$+ \frac{df(x_0,y_0,z_0)}{dy}(y-y_0) + \frac{df}{dz}(x_0,y_0,z_0)(z-z_0)$$

[for 2 dim $f(x,y)$, this is equivalent to replacing $f(x,y)$ by its tangent <u>plane</u> at (x_0, y_0)]

Often in physics we use Taylor series to expand a function in some small physical parameter ϵ. Expanding about the <u>origin</u>, the Taylor series to 2nd order becomes

$$f(\epsilon) \approx f(0) + f'(0)\,\epsilon + \frac{1}{2} f''(0)\,\epsilon^2$$

Some useful Taylor series to work out or to remember:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

works when n is positive or negative

n does not need to be integer

ex: $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \cdots$

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots$$

Similarly, take $x \to -x$ to get (signs now alternate)

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

It is good to know, or remember how to get, ~~these~~ The ~~series~~ lowest two terms in each series.