Electric field from a disk with uniform surface charge σ

Let us look at the exact answer in different limits

exact result: $\vec{E}(z) = 2\pi\sigma z\,\hat{z}\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right]$

① $|z| \gg R$ far from the disk

we can write $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{|z|\sqrt{1+\frac{R^2}{z^2}}}$ with $\frac{R^2}{z^2} \ll 1$

we want to expand the square root for small R^2/z^2.
Use Taylor series to 1st order $f(\epsilon) \simeq f(0) + f'(0)\,\epsilon$
with $f(\epsilon) = \frac{1}{\sqrt{1+\epsilon}}$ $\Rightarrow$ $f'(\epsilon) = -\frac{1}{2}\frac{1}{(1+\epsilon)^{3/2}}$

$f(0) = 1$ $\qquad$ $f'(0) = -\frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{1+\epsilon}} \approx 1 - \frac{1}{2}\epsilon$ for small ϵ

So $\frac{1}{\sqrt{1+\frac{R^2}{z^2}}} \approx 1 - \frac{R^2}{2z^2}$

So $\vec{E}(z) \approx 2\pi\sigma z\,\hat{z}\left[\frac{1}{|z|} - \frac{1}{|z|}\left(1 - \frac{R^2}{2z^2}\right)\right]$

$= \frac{2\pi\sigma z\,\hat{z}}{|z|}\frac{R^2}{2z^2} = 2\pi\sigma R^2\,\frac{z\,\hat{z}}{2|z|^3}$

$\vec{E}(z) = \frac{Q\,z\,\hat{z}}{|z|^3}$ $\qquad$ for $z > 0$, $z = |z|$
and $\vec{E} = \frac{Q}{z^2}\hat{z}$

for $z<0$, $z=-|z|$ and $\vec{E} = -\frac{Q\hat{z}}{z^2}$

In both cases $\vec{E}(z)$ is just that of a point charge Q. This is reasonable since when $|z| \gg R$, an observer at z does not "see" that the charge is smeared out over a radius R, the charge looks point-like relative to the much larger distance z.

② $|z| \ll R$ close to the disk

Now we write $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{R\sqrt{1+\frac{z^2}{R^2}}}$

expand for small $\frac{z^2}{R^2} \ll 1$. Again use $\frac{1}{\sqrt{1+\epsilon}} \approx 1-\frac{1}{2}\epsilon$

$$\simeq \frac{1}{R}\left(1-\frac{z^2}{2R^2}\right)$$

So

$$\vec{E}(z) = 2\pi\sigma z\hat{z}\left[\frac{1}{|z|} - \frac{1}{R}\left(1-\frac{z^2}{2R^2}\right)\right]$$

$$= 2\pi\sigma z\hat{z}\left[\frac{1}{|z|} - \frac{1}{R} + \frac{z^2}{2R^3}\right]$$

$$= 2\pi\sigma\frac{z}{|z|}\hat{z}\left[1 - \frac{|z|}{R} + \frac{|z|^3}{2R^3}\right]$$

for $\frac{|z|}{R} \ll 1$ we can ignore last two terms

$$\vec{E}(z) \simeq 2\pi\sigma\frac{z}{|z|}\hat{z}$$

For $z > 0$, $|z| = z$ and $\vec{E}(z) = 2\pi\sigma\,\hat{z}$

For $z < 0$, $|z| = -z$ and $\vec{E}(z) = -2\pi\sigma\,\hat{z}$

so $\vec{E}(z)$ ~~points~~ is constant (independent of z) and points perpendicularly away from the disk.

The $|z| \ll R$ limits holds either when

1) $|z|$ gets very small compared to a finite R

or

2) any $|z|$ when $R \to \infty$. This is the case of an infinite flat charged plane.

We will later use another method to recover this result for an <u>infinite</u> charged plane

$$\vec{E} = \begin{cases} 2\pi\sigma\,\hat{z} & z > 0 \\ -2\pi\sigma\,\hat{z} & z < 0 \end{cases}$$

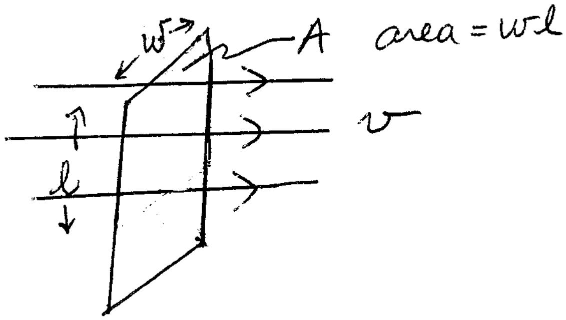
For a disk of finite R, the above results only hold along the z axis going through the center of the disk.

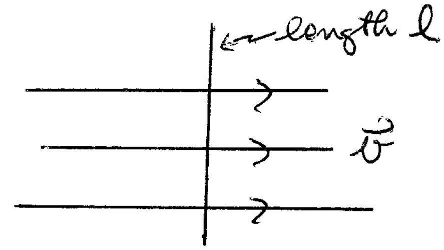
For an infinite plane, the result above holds anywhere - not just on the z axis. This is because one the disk becomes an infinite plane, there is nothing unique about where one puts the origin of the coordinate system.

<u>Flux</u> of a vector field through a surface

Imagine a fluid flowing with a constant uniform velocity $\vec{v}$. The density of particles per unit volume in the fluid is n. We want to know how many particles per unit time will flow through some given area A. This is called the "flux" of fluid through area A. We denote the flux by Φ.

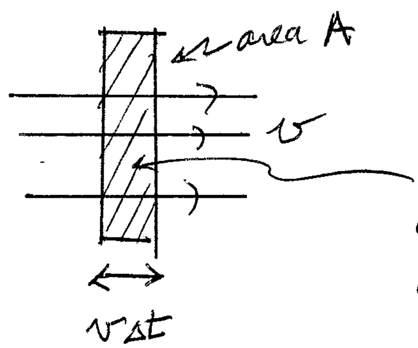
Suppose the area A is a rectangle oriented so that $\vec{v}$ is normal to the surface of A





side view - area A seen edge on

The number of particles N that flow through A in a small time Δt will be all the particles that lie in a volume $(v\Delta t)A$ as shown below

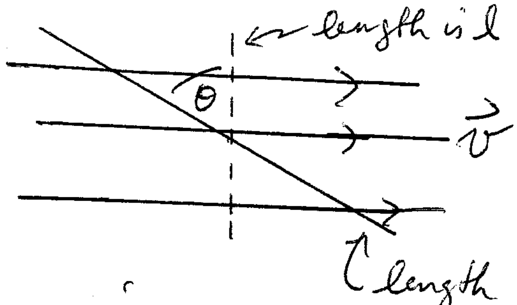


all particles in this volume at $t=0$ will have flowed through area A after time Δt has elapsed

$$N = n\, v \Delta t\, A$$

flux $\Phi = \dfrac{N}{\Delta t} = n v A$ units are $\frac{1}{cm^3} \cdot \frac{cm}{s} \cdot cm^2 = \frac{1}{s}$

Now suppose we consider a different area A' tilted with respect to $\vec{v}$. We keep the width w the same, but the length l' of A' is now larger so that the projection of l' onto the direction perpendicular to $\vec{v}$ is equal to l. See diagram below.



side view – area A' seen edge on

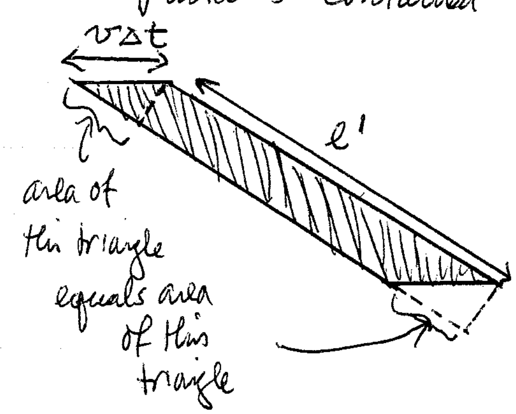
$$l' \cos\theta = l$$

$$\text{area } A' = w l' = \frac{w l}{\cos\theta}$$

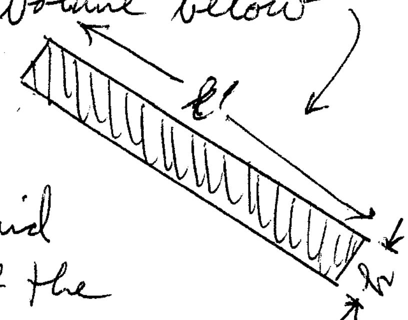
What is the flux through area A'.

It may seem obvious that the flux of particles Φ' through the area A' is the same as the flux of particles Φ through area A. But let us show explicitly that this is so.

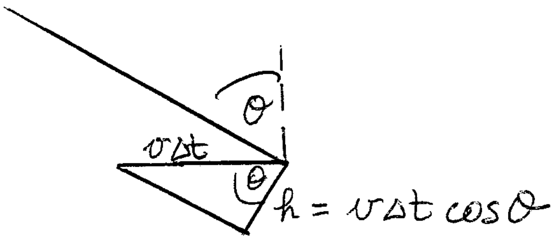
The number of particles flowing through A' in a time Δt is all the particles contained in the volume shaded below. (rhomboidal)



This is the same volume as the rectangular volume below



we need to find the height of the rectangle h, in terms of $v\Delta t$



volume is thus

$$l'wh = l'w\,v\Delta t\cos\theta$$
$$= \left(\frac{l}{\cos\theta}\right) w\,v\Delta t\cos\theta$$
$$= lw\,v\Delta t$$

number of particles flowing through A' in time Δt is therefore

$$N' = n\,lw\,v\Delta t = nAv\Delta t = N$$

so

$$\Phi' = \frac{N}{\Delta t}, \qquad \Phi = \frac{N}{\Delta t} \qquad \Rightarrow \quad \Phi' = \Phi$$

flux through A is same as flux through A' even though area of A' is larger than area of A.

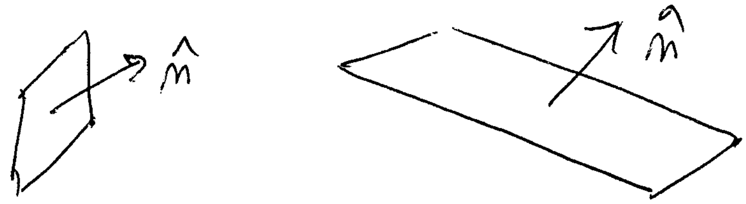
What is important is the projection of area A' perpendicular to the velocity $\vec{v}$.

We can write

$$\Phi = nvA = nvA'\cos\theta$$

reminds one of dot product

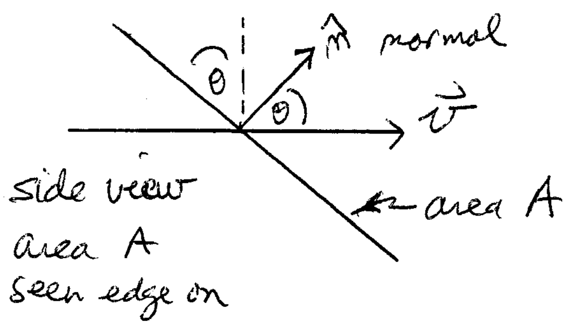
Formalize this result as follows. For any flat surface A, define the unit vector normal to the surface to be $\hat{n}$. The direction of $\hat{n}$ points from "inside" to "outside" ie the direction in which one is computing the flux through the surface.



Define the <u>area vector</u> by $\vec{A} = A\,\hat{n}$

The flux of particles through A is then

$$\Phi = n\,\vec{v}\cdot\vec{A} = n\,|\vec{v}|\,A\cos\theta$$



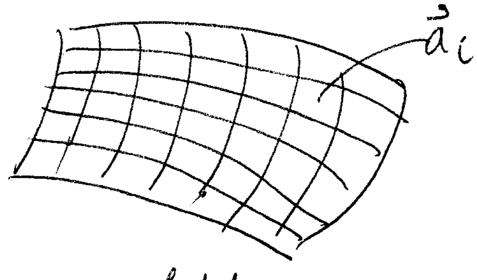
Similarly the flux of <u>any</u> vector field, for example the electric field $\vec{E}$, through a surface $\vec{A}$ is

$$\boxed{\Phi = \vec{E}\cdot\vec{A}}$$

The above hold when the vector field is spatially uniform and the area $\vec{A}$ is flat.

But we can generalize to the case where the vector field varies in space and A is any arbitrary curved surface.

Consider an arbitrary surface S which could be closed (like the surface of a sphere) or open (like a plane).



$|\vec{a}_i|$ = area of tile i

$\vec{a}_i$ points normal to S

Imagine ~~tiling~~ tiling the surface S into very small tiles $\vec{a}_i$ (direction of $\vec{a}_i$ is normal to surface S at position of tile i)

If the tiles are taken infinitesimally small they will be approximately flat. And if they are infinitesimally small, any vector field $\vec{E}$ will be approximately constant over the area a_i

We thus define the flux of $\vec{E}$ through tile $\vec{a}_i$ as

$$\Phi_i = \vec{E}_i \cdot \vec{a}_i$$

← area vector of tile i

↑ value of $\vec{E}$ on tile i

The total flux Φ through the surface S is the sum of the fluxes through ~~each~~ each tile

$$\Phi = \sum_i \Phi_i = \sum_i \vec{E}_i \cdot \vec{a}_i$$

In the limit that the tiles become infinitesimally small, this becomes the definition of a

vector surface integral

$$\Phi = \int_S \vec{E} \cdot d\vec{a}$$

differential vector area $\quad d\vec{a} = \hat{n}\, da$

vector field evaluated at positions $\vec{r}$ on surface S

integrate over surface S

When the surface S is closed, it is customary to denote this by adding a circle over the integral sign

$$\Phi = \oint_S \vec{E} \cdot d\vec{a}$$

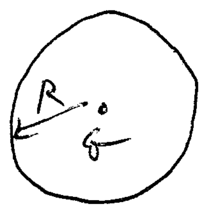
closed surface

For a general vector field $\vec{E}(\vec{r})$ and an arbitrary shaped surface S, such surface integrals can be terrible to try and calculate analytically.

However in certain simple cases of high symmetry these integrals can be done relatively easily. In this class we will see only the simple cases!

<u>Gauss' Law</u> for electrostatics

Consider a point charge q. Choose coordinate system so that q is located at the origin. Let us compute the flux of the electric field $\vec{E}$ from q through the surface of a sphere of radius R that is centered on q.



$$\vec{E} = \frac{q}{r^2}\hat{r}$$

r is radial distance outward from q. $\hat{r}$ is unit vector in radial direction

We want $\Phi = \oint_S \vec{E}\cdot d\vec{a}$

$S \leftarrow$ surface of sphere of radius R

$$\Phi = \oint_S \frac{q}{r^2}\,\hat{r}\cdot d\vec{a}$$

Now $d\vec{a}$ points in the direction normal to the surface of the sphere – this is always the outward radial direction, $\hat{n} = \hat{r}$. (Note, the direction of $\hat{r}$ varies as one varies one's position on the surface). Thus $d\vec{a}$ always points in the same direction as $\vec{E}$. Hence the vector dot product is just the same as the scalar product of the vector magnitudes

$$\vec{E} \cdot d\vec{a} = E\,da$$

Moreover, the magnitude of $\vec{E}$ at every point on the surface of the sphere is the same

$$|\vec{E}| = E = \frac{q}{R^2}$$

Thus $$\Phi = \oint_S \vec{E} \cdot d\vec{a} = \oint_S E\,da = \frac{q}{R^2}\oint_S da$$

where the last integral is just the surface area of a sphere of radius R,

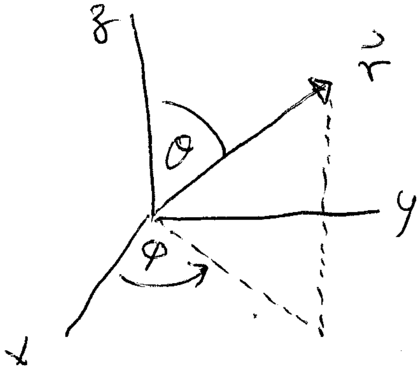
$$\oint_S da = 4\pi R^2$$

So $$\Phi = \frac{q}{R^2}\oint_S da = \frac{q}{R^2} 4\pi R^2 = \boxed{4\pi q = \Phi}$$

Note, the flux Φ is independent of the radius of the sphere R. This is because Coulomb's law is an inverse square law. The magnitude E decreases as $1/r^2$, but the surface area of the sphere increases as r^2, so the flux Φ is independent of radius of the sphere.

Next we want to show that the flux of $\vec{E}$ through any closed surface containing the charge q is also just equal to $4\pi q$

But first we review <u>spherical coordinates</u>



$r = |\vec{r}|$

θ = angle of $\vec{r}$ with respect to z axis

φ = angle of [projection of $\vec{r}$ into the xy plane] with respect to x axis

Transformation between cartesian and spherical coordinates:

$$\begin{cases} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{cases}$$

r can go from 0 to ∞

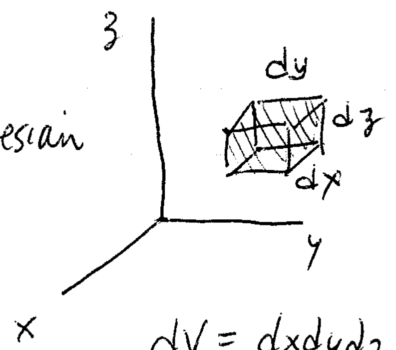
θ goes from 0 to π

φ goes from 0 to 2π

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \varphi = \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) \end{cases}$$

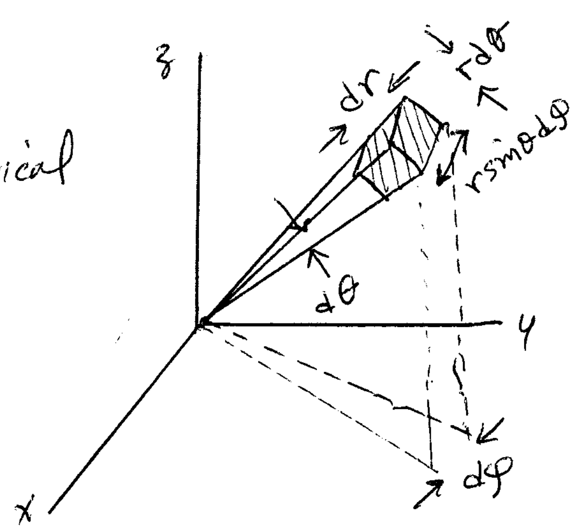
volume element

Cartesian



$dV = dx\,dy\,dz$

spherical



$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\varphi) = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

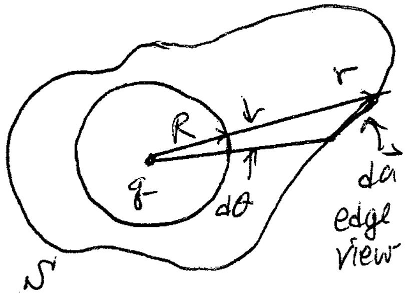
Volume of a sphere of radius R

$$V = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta \;=\; 2\pi \int_0^R dr \int_0^\pi d\theta \; r^3 \sin\theta$$

$$= 2\pi \left[-\cos\theta\right]_0^\pi \int_0^R dr\, r^2 \;=\; 4\pi \left[\frac{r^3}{3}\right]_0^R \;=\; \frac{4}{3}\pi R^3$$

area of surface of sphere of radius R

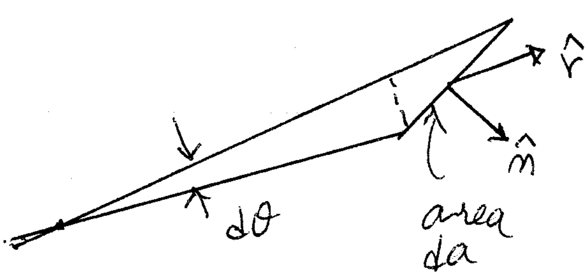
$$A = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; R^2 \sin\theta \;=\; 4\pi R^2$$



Consider now the flux of $\vec{E}$ through some arbitrary ~~closed~~ surface S as shown to left. Consider the flux through a small element of area $d\vec{a}$ spanned by the spherical angles $d\theta$ and $d\varphi$. Let r be the distance from q to $d\vec{a}$.

The flux through $d\vec{a}$ is

$$\vec{E} \cdot d\vec{a} = \frac{q}{r^2}\, \hat{r} \cdot d\vec{a}$$



$\hat{r} \cdot d\vec{a}$ is just the projection of the area da onto the direction perpendicular to $\hat{r}$, ie onto the surface of a sphere of radius r.

Thus $\hat{r} \cdot d\vec{a} = r^2 \sin\theta \, d\theta \, d\varphi$

$$\vec{E} \cdot d\vec{a} = \frac{q}{r^2}\, \hat{r} \cdot d\vec{a} = \frac{q}{r^2}\, r^2 \sin\theta \, d\theta \, d\varphi = q \sin\theta \, d\theta \, d\varphi$$

$$\vec{E}\cdot d\vec{a} = q \sin\theta \, d\theta \, d\varphi$$

Compare this to the flux of $\vec{E}$ through the corresponding element of area on an inscribed sphere of radius R that is spanned by the same $d\theta$ and $d\varphi$.

Here

$$\vec{E} = \frac{q}{R^2}\hat{r} \quad \text{and} \quad d\vec{a} = (R^2 \sin\theta \, d\theta d\varphi)\hat{r}$$

$$\vec{E}\cdot d\vec{a} = \left(\frac{q}{R^2}\right)(R^2 \sin\theta \, d\theta d\varphi) = q \sin\theta \, d\theta \, d\varphi$$

the flux is the same as through the corresponding $d\vec{a}$ on S' !

As we vary the angles θ and φ, the flux through the corresponding $d\vec{a}$'s on S' and the sphere of radius R will always be equal. Integrating over all θ and φ we thus conclude that the flux through S' is the same as the flux through the sphere

$$\Phi = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \; q \sin\theta = 4\pi q$$

So the flux through <u>any</u> closed surface containing a point charge q is

$$\oint_{S'} \vec{E}\cdot d\vec{a} = 4\pi q .$$

Suppose there are several charges q_i contained inside the surface S? By superposition the total $\vec{E}$ is the sum of the $\vec{E}_i$ from each q_i

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

Hence

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S \vec{E}_1\cdot d\vec{a} + \oint_S \vec{E}_2\cdot d\vec{a} + \cdots$$

$$= 4\pi q_1 + 4\pi q_2 + \cdots$$

$$= 4\pi (q_1 + q_2 + \cdots)$$

$$= 4\pi Q_{total}$$

$Q_{total} = \sum_i q_i$ total charge contained inside surface S

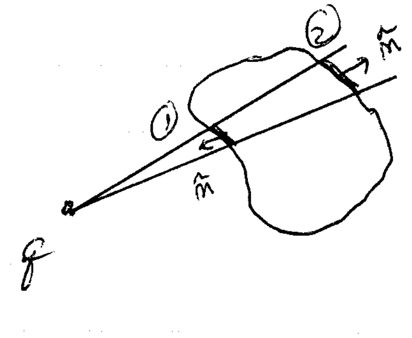
<u>Gauss' Law</u>

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = 4\pi Q}$$

total charge contained <u>inside</u> the <u>closed</u> surface S

True for any surface S, and any configuration of charge – even from continuous charge distribution $\rho(\vec{r})$.

<u>Note:</u> Flux through a closed surface S due to a charge <u>outside</u> the surface is <u>zero</u>



similarly to our previous argument the flux from q through area ② will be the same in magnitude as the flux through area ①, but it is opposite directed, so the sum from ①+② cancel.