Applying Gauss' Law

① Spherical charge distribution
$\rho(\vec{r})$ depends only on $r = |\vec{r}|$ distance from origin.

Then, since ρ specifies no preferred orientation, it must be true that $\vec{E}$ also has spherical symmetry
$\Rightarrow \vec{E}$ points only in radial direction $\hat{r}$, and can vary only with radial distance

$$\Rightarrow \vec{E}(\vec{r}) = E(r)\hat{r}$$

If we chose a spherical surface of radius r for integration in Gauss' Law we have

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S E(r)\hat{r}\cdot \underbrace{\hat{r}\,da}_{d\vec{a}}$$

since outward normal to S is always $\hat{r}$

$$= E(r)\oint_S da$$

since $E(r)$ is constant on S and $\hat{r}\cdot\hat{r} = 0$

$$= \underbrace{4\pi r^2}_{\text{area of sphere of radius } r} E(r) = 4\pi Q_{encl}$$

by Gauss' Law

total charge enclosed by S

So $\vec{E}(\vec{r}) = \dfrac{Q_{encl}}{r^2}\,\hat{r}$

$$Q_{encl} = \int_0^r dr' \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, r'^2 \sin\theta \, \rho(r')$$

$$Q_{encl} = 4\pi \int_0^r dr' \, r'^2 \rho(r')$$

If $\rho(r)$ is such that $\rho = 0$ for $r > R$, then

for all $r > R$, we get,

$$\vec{E}(\vec{r}) = \frac{Q}{r^2} \hat{r}$$

where $Q = 4\pi \int_0^R dr \, r^2 \rho(r)$ is the total charge.

So outside the charge distribution, $\vec{E}$ looks exactly like a point charge with all the charge Q at the origin.

If one is making measurements of electric field away from the charge density ρ, ie at $r > R$, there is no way one can tell that the charge causing $\vec{E}$ is smeared out on length scale R or if it is a point charge. To see that one does not have a point charge one needs to measure $\vec{E}$ at the small distances over which the charge is smeared out.

If one has a particle with charge q, and it is smeared out over a radius R, one can try to detect

R by scattering another charged particle ~~q~~, say an electron of charge e, off of q. The electron has to have enough energy to get within R of q, in order to see that q is not a point particle. The electron must thus be sent in with an initial kinetic energy $\sim \frac{eq}{R}$ ~~which is thereso~~

If $\rho(r) = 0$ for $r < r_0$, then one has for $r < r_0$ $Q_{encl} = 0$ and so $\vec{E} = 0$.

Example: spherical shell with uniform surface charge σ at radius R.

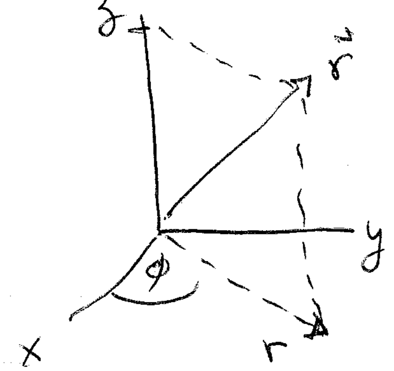
$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{4\pi R^2 \sigma}{r^2}\,\hat{r} & r > R \\ 0 & r \leq R \end{cases}$$

② Infinite straight line charge

λ uniform line charge

By symmetry $\vec{E}(\vec{r})$ must point radially outward from the line and can vary only with the radial distance from the line

Digression: Cylindrical coordinates



$\vec{r} = (r, \phi, z)$

↑ cylindrical radial coord

$r \neq |\vec{r}|$

$|\vec{r}| = \sqrt{r^2 + z^2}$

r is length of projection of $\vec{r}$ into xy plane

ϕ is angle of this projection with respect to $\hat{x}$

z is projection of $\vec{r}$ onto z axis

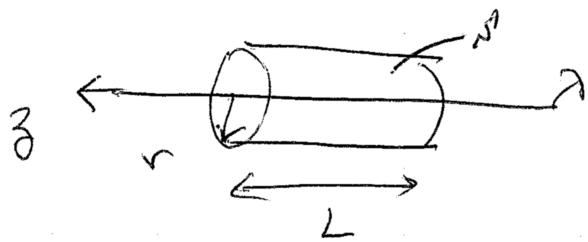
cartesian		cylindrical
z	$=$	z
x	$=$	$r\cos\phi$
y	$=$	$r\sin\phi$

cylindrical		cartesian
z	$=$	z
r	$=$	$\sqrt{x^2+y^2}$
ϕ	$=$	$\arctan\left(\frac{y}{x}\right)$

differential volume element $dV = r\,d\phi\,dr\,dz$

$$\int dx \int dy \int dz = \int dr\, r \int dz \int d\phi$$

line charge: choose as Gaussian surface a cylinder of radius r and length L



choose ~~axis of~~ line to lie along z axis

On the circular sides of S, outward normal is $+\hat{z}$ on left and $-\hat{z}$ on right. Since $\vec{E} = E(r)\,\hat{r}$ and $\hat{z} \cdot \hat{r} = 0$, the contribution to $\oint_S \vec{E} \cdot d\vec{a}$ will vanish from these sides.

On ~~From~~ the side of length L the outward normal is everywhere just $\hat{r}$, so $\vec{E} \cdot d\vec{a} = E(r)\,\hat{r} \cdot \hat{r}\, da = E(r)\,da$ on this side. So

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\substack{\text{side of}\\ \text{length } L}} E(r)\,da = E(r) \int da$$

E is const on this surface

$$= E(r) \underbrace{2\pi r L}_{\text{area of surface}} = 4\pi Q_{encl}$$

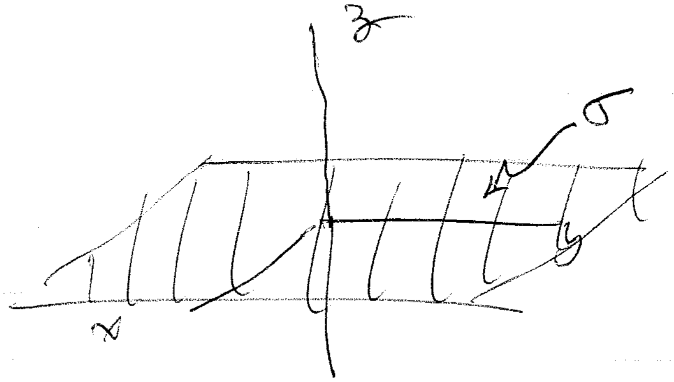
charge enclosed by S

So $E(r) = \dfrac{2\,Q_{encl}}{rL}$

with $Q_{encl} = \lambda L$ so

$$\boxed{\vec{E}(\vec{r}) = \frac{2\lambda}{r}\,\hat{r}}$$

③ infinite flat plane with uniform surface charge σ



choose coordinates so charged plane is the xy plane at $z=0$.

By symmetry $\vec{E}$ must point in $\hat{z}$ direction and can only vary with z coordinate. $\vec{E}(\vec{r}) = E(z)\hat{z}$

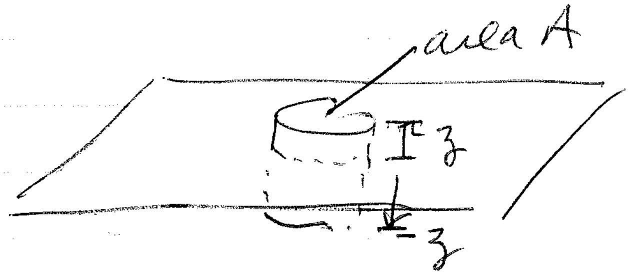
Also expect $E(z) = -E(-z)$

If $\sigma > 0$ then $\vec{E}$ points away from plane so $\vec{E}$ along $+\hat{z}$ for $z>0$ and $\vec{E}$ along $-\hat{z}$ for $z<0$.

If $\sigma < 0$ then $\vec{E}$ points towards the plane so $\vec{E}$ along $-\hat{z}$ for $z>0$ and $\vec{E}$ along $+\hat{z}$ for $z<0$.

In both cases $E(z) = -E(-z)$

For the surface of integration in Gauss Law we can take a cylinder of length $2z$ centered on the charged plane, with cross sectional area A.



Only the top and bottom sides with area A will contribute to $\oint \vec{E}\cdot d\vec{a}$ since $\vec{E} = E(z)\hat{z}$ is $\perp$ $\hat{n}$ on the other side.

$$\oint_S \vec{E}\cdot d\vec{a} = \underset{\substack{\text{top} \\ \text{side}}}{\int} E(z)\hat{z}\cdot \underbrace{\hat{z}\,da}_{\substack{d\vec{a} \\ \hat{n}=\hat{z}}} + \underset{\substack{\text{bottom} \\ \text{side}}}{\int} E(z)\,\hat{z}\cdot \underbrace{(-\hat{z}\,da)}_{\substack{d\vec{a} \\ \hat{n}=-\hat{z}}}$$

$$= E(z)A - E(-z)A$$

topside is at height z — bottom side is at height $-z$

$$= 2E(z)A \qquad \text{since } -E(-z) = E(z)$$

$$= 4\pi Q_{encl} \qquad \text{with } Q_{encl} = \sigma A$$

So

$$2E(z)A = 4\pi\sigma A$$

$$E(z) = 2\pi\sigma \qquad z>0$$

$$\vec{E}(\vec{r}) = \begin{cases} 2\pi\sigma\,\hat{z} & z>0 \\ -2\pi\sigma\,\hat{z} & z<0 \end{cases} \qquad \text{since } E(-z) = -E(z)$$

Same answer as we got from circular charged disk of radius R in the limit $z \ll R$.

Note: Both the infinite plane with σ, and the spherical shell with σ, had the following behavior in $\vec{E}$.

plane: $\vec{E}_{above} = 2\pi\sigma\hat{z}$ $\quad \vec{E}_{below} = -2\pi\sigma\hat{z}$

$$\Delta\vec{E} = \vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\hat{z}$$

Spherical shell: $\underbrace{\vec{E}_{above}}_{\text{just outside}} = \dfrac{4\pi R^2\sigma}{R^2}\hat{r}$ $\quad \underbrace{\vec{E}_{below}}_{\text{just inside}} = 0$

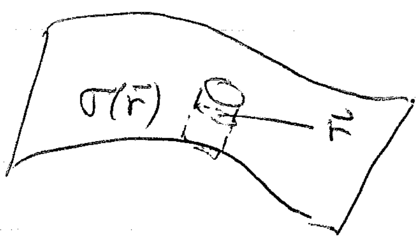
$$\Delta\vec{E} = \vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\hat{r}$$

In both cases $\vec{E}$ has a discontinuity as one passes through the charged surface. If one computes the size of the discontinuity, then in both cases

$$\Delta\vec{E} = \underbrace{\vec{E}_{above}}_{\text{just above surface}} - \underbrace{\vec{E}_{below}}_{\text{just below surface}} = 4\pi\sigma\hat{n}$$

$\hat{n}$ outward normal to surface

This is in fact a general result for any charged surface, even if σ is not uniform.



We do similarly to what we did for the flat plane. At a point $\vec{r}$ on the surface take a small cylinder with

Side view

crossectional area dA and length dl.

We first let $dl \to 0$ so top side is just above charged surface, bottom side is just below surface. Also as $dl \to 0$ the contrib from the side of length dl to $\oint \vec{E} \cdot d\vec{a}$ will vanish.

Then $\oint \vec{E} \cdot d\vec{a} = \int_{\text{above}} \vec{E} \cdot d\vec{a} + \int_{\text{below}} \vec{E} \cdot d\vec{a}$

as $dA \to 0$ we get

$$\oint \vec{E} \cdot d\vec{a} = \vec{E}_{above} \cdot d\vec{A}_{above} + \vec{E}_{below} \cdot d\vec{A}_{below}$$

$$= \vec{E}_{above} \cdot \hat{n}\, dA - E_{below} \cdot \hat{n}\, dA$$

$$= (\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{n}\, dA = 4\pi Q_{encl} = 4\pi \sigma\, dA$$

$$\Rightarrow \boxed{(\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{n} = 4\pi\sigma}$$

So component of $\vec{E}$ normal to surface has a discontinuous jump equal to $4\pi\sigma$.

To complete our proof that $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\hat{n}$ we still have to show that the tangential component of $\vec{E}$ is continuous. We will show that soon.