Force on a charged surface

The surface charge density σ will ~~???~~ produce an electric field. The electric field exerts a force back on σ. What is the force on the charged surface?

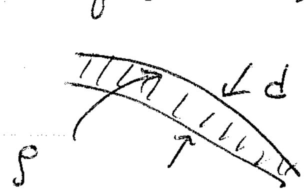
One might expect Coulomb's law to give the force per unit area

$$\vec{f} = \vec{E}\sigma$$

But at a charged surface, ~~at~~ $\vec{E}$ is discontinuous! What value of $\vec{E}$ does one use? $\vec{E}_{above}$? $\vec{E}_{below}$? The force is given by the <u>average</u> $\frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$.

$$\vec{f} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})\sigma$$

The reason there is a "problem" is because a surface charge creates a discontinuous $\vec{E}$, thus creating a question what value of $\vec{E}$ to use at the surface. One way to get around this is to smear the surface charge out over a finite small width d, such that the resulting volume has a charge density ρ with $\rho d = \sigma$. The $\vec{E}$ from this ρ is smooth and continuous and so one can then compute

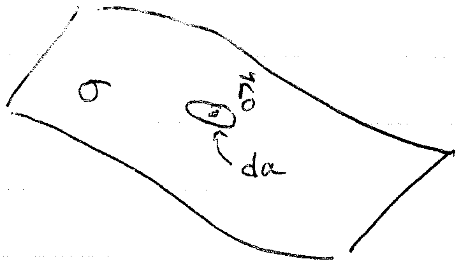


$$\vec{f} = \int dx\, \vec{E}\rho$$

↑ integrates across thickness of layer

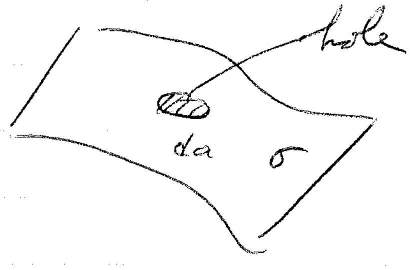
See the text sec 1.14 to see the solution worked out with this approach.

Here we will derive it a different way.



Consider the force on the small circular patch da located at position $\vec{r}_0$ on the surface. The force on this patch should be from all the charges ~~with~~ on the surface excluding those on the patch itself (since charges cannot exert a net ~~force~~ on themselves by Newton's 3rd law)

The force on the patch should be $\vec{E}_0\, \sigma(\vec{r}_0)da$ where $\vec{E}_0$ is the electric field at $\vec{r}_0$ in the center of the patch, due to all the charges not on the patch, ie from the charged surface with a small hole of area da cut out



By superposition $\vec{E}_0$ can be viewed as the sum of $\vec{E}_S$ from the entire surface with $\sigma(\vec{r})$ (including the patch) + E_p from a circular disk with surface charge $-\sigma(\vec{r}_0)$ (canceling the $+\sigma$ on the surface to create an effective hole). The hole should be very small on the length scale that $\sigma(\vec{r})$ varies so that we can take the $-\sigma(\vec{r}_0)$ on the disk to be constant (ie spatially uniform)

We know that $\vec{E}_{S\,\text{above}} = \vec{E}_{S\,\text{below}} + 4\pi\sigma(\vec{r})\,\hat{n}$

$\uparrow$ outward normal

σ at position $\vec{r}$

For the point $\vec{r}_0$ on the circular disk, we can apply our result for $\vec{E}$ from a uniformly charged disk. Since $\vec{r}_0$ always lies exactly on the disk, then no matter how small is the size of the disk we are always in the "close" limit where

$$\vec{E}_{p\,\text{above}} = -2\pi\sigma\,\hat{n}\ ,\quad \vec{E}_{p\,\text{below}} = +2\pi\sigma\,\hat{n}$$

(signs follow since charge on disk is $-\sigma$)

Now we can add $\vec{E}_S + \vec{E}_p = \vec{E}_0$

just <u>above</u> the surface we have

$$\vec{E}_{0\,\text{above}} = \vec{E}_{S\,\text{above}} - 2\pi\sigma\,\hat{n}$$

just <u>below</u> the surface we have

$$\begin{aligned}\vec{E}_{0\,\text{below}} &= \vec{E}_{S\,\text{below}} + 2\pi\sigma\,\hat{n}\\ &= \vec{E}_{S\,\text{above}} - 4\pi\sigma\,\hat{n} + 2\pi\sigma\,\hat{n}\\ &= \vec{E}_{S\,\text{above}} - 2\pi\sigma\,\hat{n}\end{aligned}$$

same as above surface

So $\vec{E}_0(\vec{r}_0)$ at the hole is <u>continuous</u> (as it should be since there is no charged surface at the hole). It does not matter if we compute it at $\vec{r}_0$ just above or just below the surface at S.

So we can then write

$$\vec{E}_0 = \frac{1}{2}\left(\vec{E}_{0\,\text{above}} + \vec{E}_{0\,\text{below}}\right)$$ since $\vec{E}_{0\,\text{above}}$ and $\vec{E}_{0\,\text{below}}$ are equal

$$= \frac{1}{2}\left(\vec{E}_{s'\,\text{above}} - 2\pi\sigma\hat{n} + \vec{E}_{s\,\text{below}} + 2\pi\sigma\hat{n}\right)$$

$$\vec{E}_0 = \frac{1}{2}\left(\vec{E}_{s\,\text{above}} + \vec{E}_{s\,\text{below}}\right)$$ averaged fields above + below surface

So force per area ~~on~~ a surface with surface charge σ is:

$$\vec{f} = \frac{1}{2}\sigma\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$$

For a conducting spherical shell of radius R

$$\vec{E}_{\text{above}} = \frac{4\pi R^2\sigma}{R^2}\hat{r} = 4\pi\sigma\hat{r}$$

$$\vec{E}_{\text{below}} = 0$$

So

$$\vec{f} = \frac{1}{2}\sigma\left(4\pi\sigma\hat{r} + 0\right) = 2\pi\sigma^2\hat{r}$$

~~Similarly, for a flat infinite plane~~

$$\cancel{\vec{E}_{\text{above}} = 2\pi\sigma\hat{n}}$$

$$\cancel{\vec{E}_{\text{below}} = -2\pi\sigma\hat{n}}$$

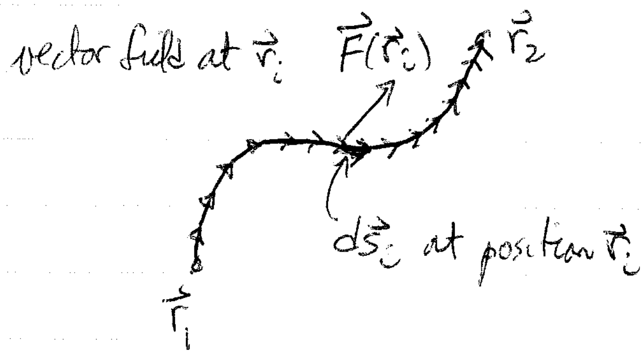
see also Purcell prob 1.29

Line Integrals

Let C be a path going from position $\vec{r}_1$ to $\vec{r}_2$.
Let $\vec{F}(\vec{r})$ be a vector field. The <u>line integral</u>

$$\int_{\substack{\vec{r}_1 \\ C}}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{s}$$

is obtained in principle by dividing C into many infinitesimally small line segments $d\vec{s}_i$ and summing $\sum_i \vec{F}(\vec{r}_i) \cdot d\vec{s}_i$ along the path



The line integral results in the limit that each $d\vec{s}_i$ becomes infinitesimally small.

In practice it may be hard to analytically evaluate $\int_{\substack{\vec{r}_1 \\ C}}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{s}$ except over particularly simple paths

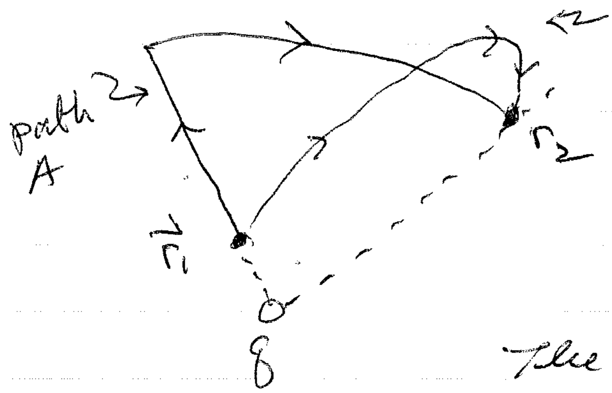
If the path C is a closed loop, ie $\vec{r}_1 = \vec{r}_2$, one denotes the line integral by

$$\oint_C \vec{F} \cdot d\vec{s}$$

$$\oint \vec{F} \cdot d\vec{s} = -\oint \vec{F} \cdot d\vec{s}$$

The <u>direction</u> one goes around the loop determines the overall sign of the integral

Consider line integral of $\vec{E}$ from a point charge q from some $\vec{r}_1$ to $\vec{r}_2$



Consider $\underset{\text{path A}}{\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{s}}$ vs $\underset{\text{path B}}{\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{s}}$

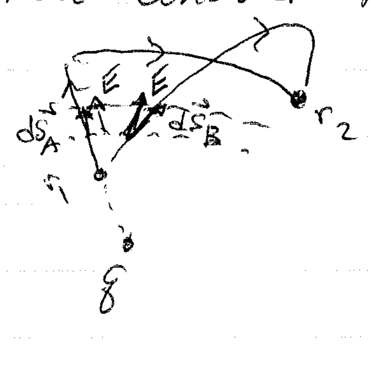
The line integral over path A is easy to do. It consists of an outward radial segment, followed by a segment at constant distance r_2 from q. Call these segments ① and ②. Since $\vec{E}(\vec{r}) = \frac{q}{r^2}\hat{r}$ along ① we have $\vec{E} \parallel d\vec{s}$. Along ② we have $\vec{E} \perp d\vec{s}$ hence $\vec{E}\cdot d\vec{s} = 0$. Therefore

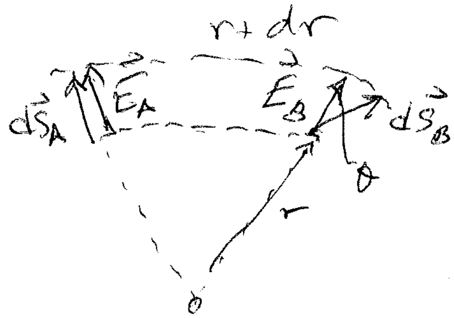
$$\underset{\text{path A}}{\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{s}} = \underset{①}{\int_{r_1}^{r_2}} \frac{q}{r^2}\,dr + \underbrace{0}_{\text{contribution from ②}}$$

$$= \left[-\frac{q}{r}\right]_{r_1}^{r_2} = \frac{q}{r_1} - \frac{q}{r_2}$$

$$= q\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Now consider the integral over path B. Compare the segment of the path A that goes from radius r to $r+dr$ (this is $d\vec{s}_A$) with the corresponding segment on path B (this is $d\vec{s}_B$).





see Fig 2.2 in text for better picture!

$|\vec{E}_A| = |\vec{E}_B|$ because they are at same r

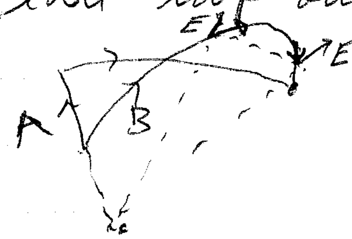
and $\vec{E}_B \cdot d\vec{s}_B = |\vec{E}_B| \underbrace{|d\vec{s}_B| \cos\theta}$

projection of $d\vec{s}_B$ on radial direction

$$|d\vec{s}_B| \cos\theta = |d\vec{s}_A|$$

so $\underset{\text{path A}}{\vec{E}_A \cdot d\vec{s}_A} = \underset{\text{path B}}{\vec{E}_B \cdot d\vec{s}_B}$

Similarly for all path segments. The segment of path B that loop out beyond distance r_2 cancels itself out



because the segments $d\vec{s}$ at the two points where B passes distance $r > r_2$ are in opposite directions.

We conclude that $\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{l}$, with $\vec{E}$ from a point charge, is independent of the path.

(Similar arguments were used to show that work done moving q_2 from ∞ to $\vec{r}_2$ in presence of q_1 at $\vec{r}_1$ is independent of the path)

Consider now any electric field $\vec{E}$ that arises from a static configuration of charges, $q_1, q_2, \cdots q_N$

If $\vec{E}_i(\vec{r})$ is the field from q_i, then $\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$

and

$$\underset{\text{path } C}{\int_{\vec{r}_1}^{\vec{r}_2}} \vec{E} \cdot d\vec{s} = \underset{\text{path } C}{\int_{\vec{r}_1}^{\vec{r}_2}} \vec{E}_1 \cdot d\vec{s} + \underset{\text{path } C}{\int_{\vec{r}_1}^{\vec{r}_2}} \vec{E}_2 \cdot d\vec{s} + \cdots$$

Since each individual $\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}_i \cdot d\vec{s}$ does not depend on what path C is taken, neither will the sum of the terms, hence neither does $\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s}$.

If we have a continuous charge distribution $\rho(\vec{r})$ then the same thing applies – the sum over the terms from each q_i now becomes an integral over terms for each $\rho(\vec{r})dV$ in the distribution $\rho(\vec{r})$.

$$\Rightarrow \boxed{\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s} \text{ is independent of the path for any \underline{electrostatic} field } \vec{E}}$$

If we let $\vec{r}_2 = \vec{r}_1$, ie have a closed path, then we must conclude

$$\boxed{\oint \vec{E} \cdot d\vec{s} = 0 \quad \text{all closed paths}}$$

Holds only for electrostatic fields

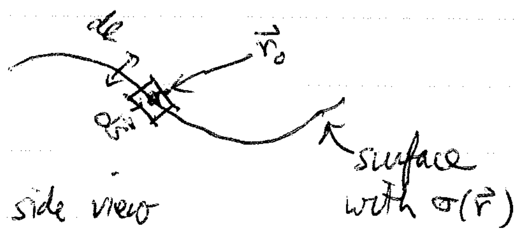
This is since we can always make an infinitesimal loop from $\vec{r}_1$ back to itself and $\oint \vec{E} \cdot d\vec{s} = 0$ on this loop vanishes because the length of the loop vanishes - Hence ~~all~~ $\oint \vec{E} \cdot d\vec{l}$ over all loops passing through $\vec{r}_1$ must vanish

Behavior of $\vec{E}$ at a charged surface

We already found that

$$\text{(1)} \qquad (\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{n} = 4\pi\sigma$$

Now consider tangential component of $\vec{E}$



consider loop as shown with $dl \to 0$

$$0 = \oint \vec{E} \cdot d\vec{s} = \vec{E}_{above}(\vec{r}_0) \cdot d\vec{s}_{above} + \vec{E}_{below}(\vec{r}_0) \cdot d\vec{s}_{below}$$

$$= \left(\vec{E}_{above}(\vec{r}_0) - \vec{E}_{below}\right) \cdot d\vec{s}_{above} = 0$$

since $d\vec{s}_{above} = -d\vec{s}_{below}$

true for all directions $d\vec{s}$ lying in plane of surface at $\vec{r}_0$

(2) $\Rightarrow$ tangential component of $\vec{E}$ is continuous at surface

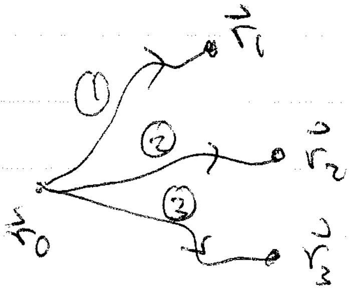
$$(1) + (2) \Rightarrow \vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$$

outward normal

Since $\int \vec{E}\cdot d\vec{s}$ is independent of the path, we can define a scalar quantity

$$\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$$

line integral starting at a fixed reference point $\vec{r}_0$, going to $\vec{r}$.



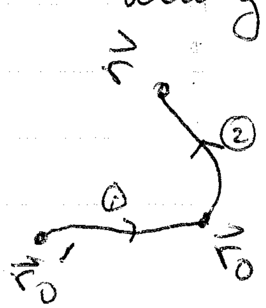
$$\phi(\vec{r}_1) = -\int_{r_0}^{\vec{r}_1} \vec{E}\cdot d\vec{s}$$
path ①

$$\phi(\vec{r}_2) = -\int_{\vec{r}_0}^{\vec{r}_2} \vec{E}\cdot d\vec{s}$$
path ②

$$\phi(\vec{r}_3) = -\int_{\vec{r}_0}^{\vec{r}_3} \vec{E}\cdot d\vec{s}$$
path ③

For each position $\vec{r}$ the reference point $\vec{r}_0$ must be kept the same. The value of $\phi(\vec{r})$ is indep of the path from $\vec{r}_0$ to $\vec{r}$.

How does ϕ change if we changed $\vec{r}_0$?
Define $\phi'(\vec{r}) = -\int_{\vec{r}_0'}^{\vec{r}} \vec{E}\cdot d\vec{s}$. Then we can always write



$$\phi'(\vec{r}) = -\int_{①+②} \vec{E}\cdot d\vec{s} = -\int_{①} \vec{E}\cdot d\vec{s} - \int_{②} \vec{E}\cdot d\vec{s}$$

$$= \phi'(r_0) + \phi(\vec{r})$$

constant for all $\vec{r}$

So $\phi'(\vec{r}) = \phi(\vec{r}) + \text{constant}$

The choice of reference point $\vec{r}_0$ only shifts the value of $\phi(\vec{r})$ by a constant for all $\vec{r}$.

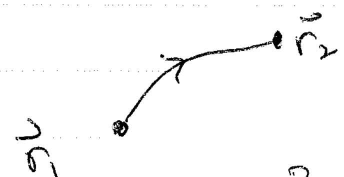
$\phi(\vec{r})$ is called the electrostatic potential for the ~~electrostatic~~ electric field $\vec{E}$.

Physical meaning of $\phi(\vec{r})$

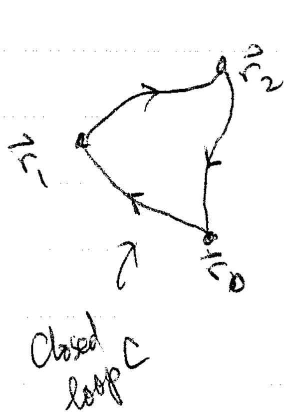
Suppose we place a charge in an electric field $\vec{E}$ at position $\vec{r}_1$, and we then move it to position $\vec{r}_2$. The work we do on the charge is

$$W = -q\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{s}$$

(the mechanical force we must exert is $-q\vec{E}$)



But since $\oint \vec{E}\cdot d\vec{s} = 0$ for all closed loops



we have $$0 = -\oint_C \vec{E}\cdot d\vec{s} = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{E}\cdot d\vec{s} - \int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{s} - \int_{\vec{r}_2}^{\vec{r}_0} \vec{E}\cdot d\vec{s}$$

$$\Rightarrow 0 = \phi(\vec{r}_1) + \frac{W}{q} + \int_{\vec{r}_0}^{\vec{r}_2} \vec{E}\cdot d\vec{s}$$

sign changed since we reverse direction of path

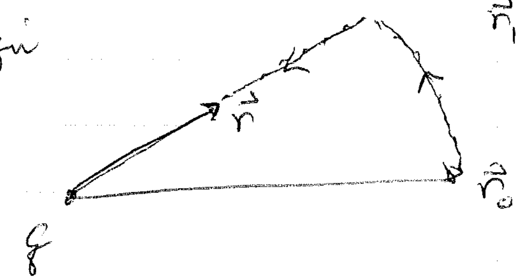
$$0 = \phi(\vec{r}_1) + \frac{W}{q} - \phi(\vec{r}_2)$$

$$\boxed{W = q\left[\phi(\vec{r}_2) - \phi(\vec{r}_1)\right]}$$

The work done to move q from $\vec{r}_1$ to $\vec{r}_2$ is just q times the potential difference $\phi(\vec{r}_2) - \phi(\vec{r}_1)$

Examples

① point charge q at origin



$$\vec{E} = \frac{q}{r^2}\hat{r}$$

$$\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$$

we can evaluate along path as shown to get

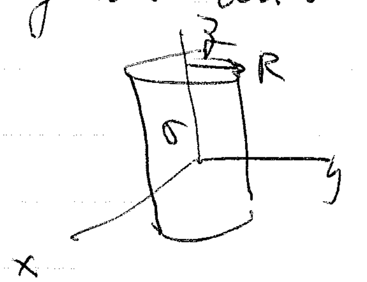
$$= -\int_{r_0}^{r} \frac{q}{r'^2}\,\hat{r}\cdot\hat{r}\,dr' = \left[\frac{q}{r'}\right]_{r_0}^{r}$$

$$= q\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

it is usual to choose reference point at $r_0 \to \infty$ in which case

$$\phi(\vec{r}) = \frac{q}{r}.$$

② cylinder with uniform surface charge σ at radius R

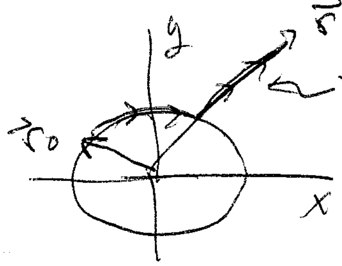


$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{2(2\pi R\sigma)}{r}\hat{r} & r > R \\ 0 & r < R \end{cases}$$

cylindrical radial coordinate

$2\pi R\sigma = \lambda$ ~~2πRσ~~ ~~this~~ charge/length

Choose $\vec{r}_0$ a point at $|\vec{r}_0| = \vec{R}$ integrate to cylindrical radius r.



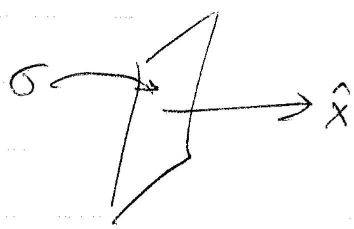
$$\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$$

$$= -\int_R^r \frac{4\pi R\sigma}{r'}\,dr'$$

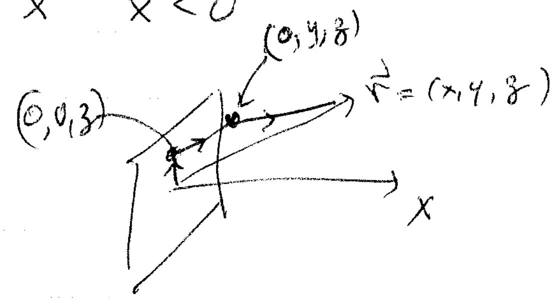
$$= -4\pi R\sigma\left[\ln r'\right]_R^r$$

$$\phi(r) = \begin{cases} -4\pi R\sigma \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

③ <u>infinite plane</u> with uniform σ



$$\vec{E} = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$



Take $\vec{r}_0 = 0$.

$$\phi(\vec{r}) = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{s}$$

$$= -\int_0^x E\,dx = -2\pi\sigma x \qquad x > 0$$

For $x < 0$, same as above but $\vec{E} = -2\pi\sigma\,\hat{x}$

so $-\int_0^x E\,dx = +2\pi\sigma x \qquad x < 0$

$$\phi(\vec{r}) = \begin{cases} -2\pi\sigma x, & x > 0 \\ +2\pi\sigma x, & x < 0 \end{cases}$$