Given $\vec{E}$, we can in principle determine ϕ via

$$\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$$

Similarly, given $\phi(\vec{r})$, we can determine $\vec{E}(\vec{r})$

Consider

$$d\phi \equiv \phi(\vec{r}+d\vec{r}) - \phi(\vec{r}) = -\int_{\vec{r}}^{\vec{r}+d\vec{r}} \vec{E}\cdot d\vec{s} = -\vec{E}\cdot d\vec{r}$$

for sufficiently small $d\vec{r}$

From calculus we have

$$\phi(\vec{r}+d\vec{r}) = \phi(x+dx, y+dy, z+dz)$$

$$= \phi(x,y,z) + \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dy$$

where partial derivatives evaluated at $\vec{r}=(x,y,z)$

So

$$\phi(\vec{r}+d\vec{r}) - \phi(\vec{r}) = \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot(dx, dy, dz)$$

$$= -\vec{E}\cdot d\vec{r} = (-E_x, -E_y, -E_z)\cdot(dx, dy, dz)$$

Since this is true for all $d\vec{r}$ infinitesimally small in length but in arbitrary direction, it must be true that

$$\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (-E_x, -E_y, -E_z)$$

or in vector notation

$$\vec{E} = -\left[\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}\right]$$

the vector

$$\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$$

is called the gradient of ϕ and is written symbolically as grad ϕ or as $\vec{\nabla}\phi$.

The symbol $\vec{\nabla}$ represents the differential operator

$$\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

so $\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$

Note

$$d\phi \equiv \phi(\vec{r}+d\vec{r}) - \phi(\vec{r}) = \vec{\nabla}\phi \cdot d\vec{r}$$

$d\phi$ is maximum when $\vec{\nabla}\phi$ and $d\vec{r}$ are parallel. Hence the direction of $\vec{\nabla}\phi$ gives the direction in which the function ϕ changes most rapidly

If $d\vec{r}$ is $\perp$ to $\vec{\nabla}\phi$, then $d\phi = 0$. Such a $d\vec{r}$ therefore points in a direction along a contour of constant ϕ. $\Rightarrow$ $\vec{\nabla}\phi$ is always $\perp$ to contours of constant ϕ.

Potential $\phi(\vec{r})$ from a continuous charge distribution $\rho(\vec{r})$

from a point charge at origin

$$\phi(\vec{r}) = \frac{q}{r} \qquad \text{(reference point } \vec{r}_0 \to \infty)$$

For a point charge q_1 at a position $\vec{r}_1$, we must replace r in the above by the distance from $\vec{r}_1$ to $\vec{r}$

$$\phi(\vec{r}) = \frac{q_1}{|\vec{r}-\vec{r}_1|}$$

For many charges q_i at positions $\vec{r}_i$

$$\phi(\vec{r}) = \sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|}$$

For a continuous charge density $\rho(\vec{r})$

$$\phi(\vec{r}) = \int_{\substack{\text{all space}\\ \text{where}\\ \rho\neq 0}} \frac{\rho(\vec{r}\,')\,dV'}{|\vec{r}-\vec{r}\,'|} = \int \frac{\rho(x',y',z')\,dx'\,dy'\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

For continuous surface charge density $\sigma(\vec{r})$

$$\phi(\vec{r}) = \int_{\text{surface}} \frac{\sigma(\vec{r}\,')\,da'}{|\vec{r}-\vec{r}\,'|}$$

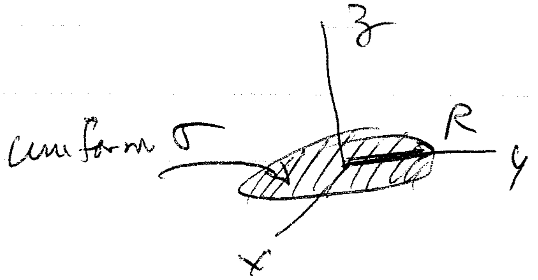
$\vec{r}\,'$ ~~vary~~ varies over surface

For a continuous line charge density

$$\phi(\vec{r}) = \int_{\text{curve}} \frac{\lambda(\vec{r}\,')\,ds'}{|\vec{r}-\vec{r}\,'|}$$

$\vec{r}\,'$ varies along line

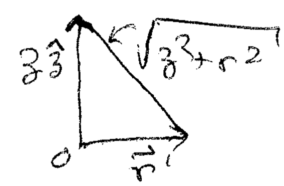
<u>Example</u> potential from a uniformly charged disk



[Note: I'll use ϕ for potential and φ for angle

for $\vec{r} = z\hat{z}$ on z axis $\quad \vec{r}\,' = r\cos\varphi\,\hat{x} + r\sin\varphi\,\hat{y}$

$$\phi(z\hat{z}) = \underbrace{\int_0^R dr\,r \int_0^{2\pi} d\varphi}_{\int da} \frac{\sigma}{\sqrt{z^2+r^2}}$$



$$= 2\pi\sigma \int_0^R dr \frac{r}{\sqrt{z^2+r^2}}$$

$$= 2\pi\sigma \left[\sqrt{z^2+r^2}\right]_0^R$$

$\sqrt{z^2} = |z|$

$$\phi(z) = 2\pi\sigma\left[\sqrt{z^2+R^2} - |z|\right]$$

If we want to comput $\vec{E} = -\vec{\nabla}\phi$ we can only compute $E_z = -\frac{\partial\phi}{\partial z}$ with the above result, because we don't know how ϕ varies with x or y if we have only

computed ϕ along the z axis, ie $\phi(0,0,z)$.
But by symmetry we know $E_x = E_y = 0$ for $\vec{r} = z\hat{z}$.

$$E_z = -\frac{\partial}{\partial z}\left\{2\pi\sigma\left[\sqrt{z^2+R^2} - |z|\right]\right\}$$

$$= -2\pi\sigma\left[\frac{z}{\sqrt{z^2+R^2}} - \frac{z}{|z|}\right]$$

since $\frac{\partial |z|}{\partial z} = \begin{cases} +1 & z>0 \\ -1 & z<0 \end{cases}$

$$E_z = 2\pi\sigma\left[\frac{z}{|z|} - \frac{z}{\sqrt{z^2+R^2}}\right]$$

$$= 2\pi\sigma z\left[\frac{1}{|z|} - \frac{1}{\sqrt{z^2+R^2}}\right]$$

which is the same as we found earlier from Coulomb's law.

Note that the computation of ϕ is in general easier than the computation of E

$$\phi(\vec{r}) = \int dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \vec{E}(\vec{r}) = \int dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\,\widehat{\vec{r}-\vec{r}'}$$

scalar integral $\qquad$ vector integral

Since ϕ involves a scalar integral while $\vec{E}$ involves a vector integral.

Potential Energy and the electrostatic potential

From before, for a set of charges $\{q_i\}$ at positions $\{\vec{r}_i\}$

$$U = \frac{1}{2} \sum_{j=1}^{N} q_j \sum_{k \neq j} \frac{q_k}{r_{jk}}$$

$$= \frac{1}{2} \sum_{j=1}^{N} q_j \, \phi^{(j)}(\vec{r}_j)$$

potential at $\vec{r}_j$ due to all charges except q_j

For a continuous distribution of charges, the above becomes

$$U = \frac{1}{2} \int_{\vec{r} \neq \vec{r}'} d^3r \int d^3r' \, \frac{\rho(\vec{r}) \, \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$d^3r = dx\,dy\,dz$
$d^3r' = dx'\,dy'\,dz'$

but for the continuous ρ, when we do double integral, the amount of space where $\vec{r} = \vec{r}'$ gives a negligible contribution

$$U = \frac{1}{2} \int d^3r \, \rho(\vec{r}) \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\boxed{U = \frac{1}{2} \int d^3r \, \rho(\vec{r}) \, \phi(\vec{r}')}$$

Divergence of a vector field

We already met the vector differential operator

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \text{ or } \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

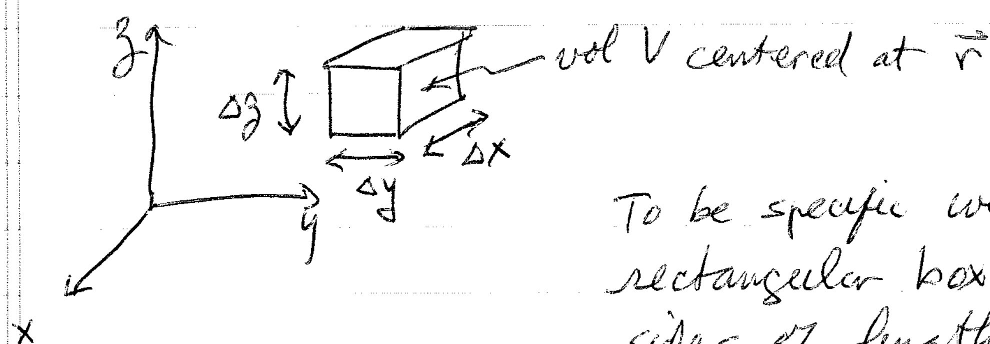
Since $\vec{\nabla}$ looks like a vector we can define its dot product with a vector field $\vec{F}(\vec{r})$

$$\vec{\nabla}\cdot\vec{F}(\vec{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(F_x, F_y, F_z\right)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

This is called the <u>divergence</u> of $\vec{F}$ and written as $\vec{\nabla}\cdot\vec{F}$ or $\operatorname{div}\vec{F}$

To find the physical meaning of the divergence, consider the flux of $\vec{F}$ through the surface S bounding an infinitesimally small volume V of space at a position $\vec{r}$.



To be specific we take a rectangular box with sides of lengths $\Delta x, \Delta y, \Delta z$

Consider the flux of $\vec{F}$ through the surface S bounding the box

$$\oint_S \vec{F}\cdot d\vec{a} = \underset{\substack{\text{top + bottom}\\ \text{sides}}}{\int} \vec{F}\cdot d\vec{a} + \underset{\substack{\text{front + back}\\ \text{sides}}}{\int} \vec{F}\cdot d\vec{a} + \underset{\substack{\text{left + right}\\ \text{sides}}}{\int} \vec{F}\cdot d\vec{a}$$

Consider the integral over the top side at $z + \frac{\Delta z}{2}$

$$\int_{top} \vec{F}\cdot d\vec{a} = \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \; F_z(x', y', z+\tfrac{\Delta z}{2})$$

F_z since $\hat{n} = \hat{z}$ on this side

Similarly

$$\int_{bottom} \vec{F}\cdot d\vec{a} = \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \left[-F_z(x', y', z-\tfrac{\Delta z}{2})\right]$$

$-F_z$ since $\hat{n} = -\hat{z}$ on this side

If the box is infinitesmally small, then F_z does not vary much over the length Δz.

For small Δz we can expand the integrand

$$F_z(x', y', z \pm \tfrac{\Delta z}{2}) \simeq F_z(x', y', z) \pm \frac{\partial F_z}{\partial z}(x', y', z)\,\frac{\Delta z}{2}$$

and so get

$$\int_{top} \vec{F}\cdot d\vec{a} + \int_{bottom} \vec{F}\cdot d\vec{a}$$

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \left[\left(F_z(x',y',z) + \frac{\partial F_z}{\partial z}(x',y',z)\frac{\Delta z}{2} \right) - \left(F_z(x',y',z) - \frac{\partial F_z}{\partial z}(x',y',z)\frac{\Delta z}{2} \right) \right]$$

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \quad \frac{\partial F_z}{\partial z}(x',y',z)\,\Delta z$$

Now if Δx and Δy are sufficiently small $\frac{\partial F_z}{\partial z}(x',y',z)$ varies negligibly over the range of integration, and so we may approximate by

$$\int_{top} \vec{F}\cdot d\vec{a} + \int_{bottom} \vec{F}\cdot d\vec{a} = \frac{\partial F_z}{\partial z}(x,y,z)\,\Delta x\,\Delta y\,\Delta z$$

Similarly

$$\int_{front} \vec{F}\cdot d\vec{a} + \int_{back} \vec{F}\cdot d\vec{a} = \frac{\partial F_x}{\partial x}(x,y,z)\,\Delta x\,\Delta y\,\Delta z$$

and

$$\int_{right} \vec{F}\cdot d\vec{a} + \int_{left} \vec{F}\cdot d\vec{a} = \frac{\partial F_y}{\partial y}(x,y,z)\,\Delta x\,\Delta y\,\Delta z$$

Adding all the terms gives

$$\oint_S \vec{F}\cdot d\vec{a} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

$$= \vec{\nabla}\cdot\vec{F}(\vec{r}) \, (\Delta x \, \Delta y \, \Delta z)$$

or $$\boxed{\vec{\nabla}\cdot\vec{F}(\vec{r}) = \frac{\oint_S \vec{F}\cdot d\vec{a}}{Vol}}$$

for infinitesimally small volume Vol bounded by surface S centered at position $\vec{r}$

Above does not depend on size of box $\Delta x, \Delta y, \Delta z$ in the limit $\Delta x, \Delta y, \Delta z \to 0$

We have proven the above for the specific case of a rectangular volume. We leave it to your math class to prove that this result remains true no matter what the shape of the volume, as long as it is infinitesmally small.

$$\vec{\nabla}\cdot\vec{F}(\vec{r}) = \lim_{V\to 0} \left[\frac{\oint_S \vec{F}\cdot d\vec{a}}{V} \right]$$

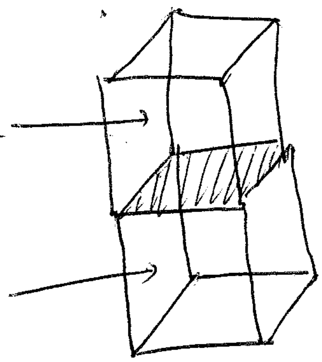
for any shape V bounded by surface S centered at position $\vec{r}$

Gauss' Theorem of Vector Calculus

Consider two rectangular boxes stacked on top of one another

box 2

box 1



let S' be the exterior sides that form the outside surface of the two boxes.

let S_1 and S_2 be the surfaces of box 1 and box 2 respectively

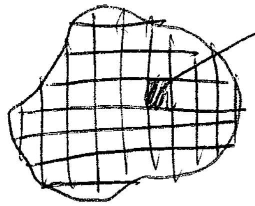
When we do $\oint_{S_1} \vec{F}\cdot d\vec{a}$ there is a contribution from the shaded side that is common with surface S'_2.
But if we add

$$\oint_{S_1} \vec{F}\cdot d\vec{a} + \oint_{S_2} \vec{F}\cdot d\vec{a}$$

the contribution from the shaded side in the $\oint_{S_1}$ exactly cancels the contribution from this side in $\oint_{S_2}$. This is because the outward normal of this side in doing $\oint_{S_2}$ points in the opposite direction from when we do $\oint_{S_1}$. Hence the contributions of this side to $\oint_{S_1}$ and $\oint_{S_2}$ are equal in magnitude but opposite in sign. So when we add them they cancel.

Thus $\oint_S \vec{F}\cdot d\vec{a} = \oint_{S_1} \vec{F}\cdot d\vec{a} + \oint_{S_2} \vec{F}\cdot d\vec{a}$

In a similar manner one can divide any finite volume V, bounded by surface S, into many infinitesmal volumes dV_i, centered at positions $\vec{r}_i$, each bounded by the surface S_i



Then,

$$\oint_S \vec{F}\cdot d\vec{a} = \sum_i \oint_{S_i} \vec{F}\cdot d\vec{a}$$

contributions from internal sides all cancel in pairs.

But as $dV_i \to 0$ we have

$$\oint_{S_i} \vec{F}\cdot d\vec{a} = dV_i \;\vec{\nabla}\cdot\vec{F}(\vec{r}_i)$$

So $\oint_S \vec{F}\cdot d\vec{a} = \sum_i dV_i \;\vec{\nabla}\cdot\vec{F}(\vec{r}_i)$

as $dV_i \to 0$ the sum becomes an integral and

$$\boxed{\oint_S \vec{F}\cdot d\vec{a} = \int_V dV \;\vec{\nabla}\cdot\vec{F}}$$ ⟸ Gauss' Theorem

Apply to Gauss Law of electrostatics

$$\oint_S \vec{E}\cdot d\vec{a} = 4\pi Q_{encl}$$

$$\Vert \qquad\qquad\qquad \Vert$$

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = 4\pi \int_V dV\, \rho$$

Since the integrals are equal for any volume of integration V, it must be true that the integrands are everywhere equal

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$$ Gauss' Law in differential form

In Cartesian coordinates

$$\frac{\partial E_x}{\partial x}(x,y,z) + \frac{\partial E_y}{\partial y}(x,y,z) + \frac{\partial E_z}{\partial z}(x,y,z) = 4\pi\rho(x,y,z)$$