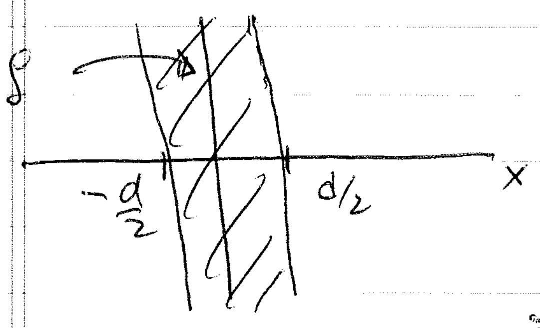
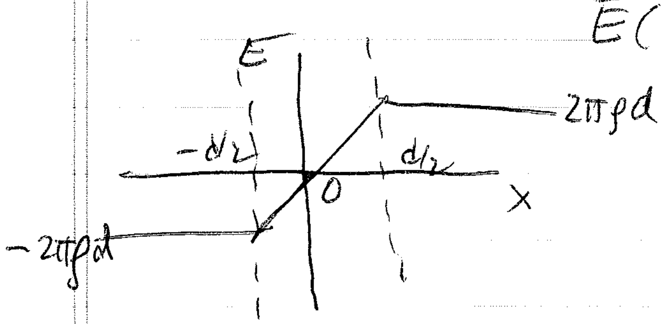
Example: Consider $\vec{E}$ from a planar slab of thickness d with uniform charge density ρ



Solution for $\vec{E}(\vec{r})$ (see solution to Purcell 1.19 on Recitation page of website) is

$$\vec{E}(\vec{r}) = \begin{cases} 2\pi\rho d\,\hat{x} & x > \frac{d}{2} \\ -2\pi\rho d\,\hat{x} & x < -\frac{d}{2} \\ 4\pi\rho x\,\hat{x} & -\frac{d}{2} < x < \frac{d}{2} \end{cases}$$



Now for $\vec{E}(\vec{r}) = E(x)\,\hat{x}$ we have

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} = \begin{cases} 0 & x > \frac{d}{2} \\ 0 & x < -\frac{d}{2} \\ 4\pi\rho & -\frac{d}{2} < x < \frac{d}{2} \end{cases}$$

so we have agreement with

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

Laplacian

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \text{also} \quad &\vec{E} = -\vec{\nabla}\phi \end{aligned}\right\} \Rightarrow -\vec{\nabla}\cdot(\vec{\nabla}\phi) = 4\pi\rho$$

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\phi) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \equiv \nabla^2\phi$$

Laplacian operator

$\vec{\nabla}\cdot\vec{\nabla} \equiv \nabla^2$ also sometimes written as Δ

$$\boxed{-\nabla^2\phi = 4\pi\rho}$$

Poisson's ~~Laplace's~~ equ for electrostatic potential

when $\rho = 0$ we have $-\nabla^2\phi = 0$

this is called Laplace's equ.

Laplace's equ $\nabla^2 \phi = 0$ holds in regions where there is no charge.

Solutions to Laplace's equ are known as <u>harmonic functions</u> They have many special properties. One is the following.

If S is the surface of a sphere $\overbrace{\text{centered}}^{\text{of radius } R}$ about a position $\vec{r}$, then

$$\boxed{\frac{1}{4\pi R^2} \oint_S \phi \, da = \phi(\vec{r})}$$ see proof in text

i.e. ϕ averaged over the surface of S is equal to ϕ at the center of the sphere. True for any radius sphere R.

One consequence of this result is that there can be <u>no stable ~~sto~~ electrostatic equilibrium</u>! at any point in empty space.

<u>Specifically</u>, suppose there is an electric field $\vec{E} = -\vec{\nabla}\phi$ that is due to some distribution of charge (or collection of point ~~g~~ charges)

Suppose $\vec{r}_0$ is a point in space at which there is no charge, and there is no charge in a finite volume of space surrounding $\vec{r}_0$, Consider putting a test charge q_0 at position $\vec{r}_0$. The work needed to do that is $W = q_0 \phi(\vec{r}_0)$

↑ potential due to all charge except q_0

If $\vec{r}_0$ is a point of stable equilibm, then $\vec{r}_0$ must be a minimum of the potential energy $U = q\phi(\vec{r})$ for some neighborhood about $\vec{r}_0$.

Draw a sphere surrounding point $\vec{r}_0$ of small radius such that it encloses none of the charge that produces ϕ. If $\phi(\vec{r}_0)$ is a minimum, then if the radius is sufficiently small, $\phi(\vec{r})$ on the surface of the sphere would always be bigger than $\phi(\vec{r}_0)$ (assuming $q_0 > 0$). But this would mean

$$\underbrace{\frac{1}{4\pi R^2} \oint_S \phi \, da}_{\text{average } \phi \text{ over the sphere}} > \phi(\vec{r}_0)$$

This violates the property of harmonic functions, and ϕ is harmonic since $\nabla^2 \phi = 0$ everywhere inside the sphere!

Hence $\phi(\vec{r}_0)$ is not a local minimum

Similarly if we use $q_0 < 0$ we conclude $\phi(\vec{r}_0)$ cannot be a local maximum.

$\Rightarrow$ A function $\phi(\vec{r})$ that is harmonic, ie $\nabla^2\phi = 0$, in a region of space $\mathbb{R}$ can have no local maximum or minimum in that region.

$\Rightarrow$ there are no points of stable electrostatic equilibria in that region

Another way to see it: If $\vec{r}_0$ was a stable equilibrium, the force on q_0 if it is displaced a small distance from $\vec{r}_0$ in any direction should always point back towards $\vec{r}_0$. Since $\vec{F} = q_0\vec{E}$ it means that $\vec{E}$ always points inwards to $\vec{r}_0$. This would mean

$$\oint_S \vec{E}\cdot d\hat{a} > 0$$

S is surface of small vol surrounding $\vec{r}_0$.

But by Gauss Law

$$\oint_S \vec{E}\cdot d\vec{a} = 4\pi Q_{encl} = 0$$

Since $Q_{encl} = 0$. Remember $\vec{E}$ is due to all charge except the test charge q_0, and space is empty of charge within in finite volume about $\vec{r}_0$.

Curl of a vector field

$\vec{\nabla}$ the vector differential operator

gradient $\vec{\nabla}\varphi = -\vec{E}$ — gradient of scalar function is a vector field

divergence $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ — divergence of a vector field is a scalar function

can also do

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(E_x, E_y, E_z)$$

called the "curl" of $\vec{E}$, or sometimes called the "circulation" of $\vec{E}$, written also curl $\vec{E}$

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

Tricks to remember the components of $\vec{\nabla}\times\vec{E}$

① determinant

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

② work it out algebraically and use right hand rule for cross products between basis vectors.

$$\vec{\nabla} \times \vec{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \times \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z \right)$$

multiply through term by term

$$= \hat{x}\frac{\partial}{\partial x} \times \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z \right)$$

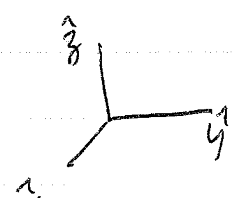
$$+ \hat{y}\frac{\partial}{\partial y} \times \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z \right)$$

$$+ \hat{z}\frac{\partial}{\partial z} \times \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z \right)$$

$$= (\hat{x}\times\hat{x})\frac{\partial E_x}{\partial x} + (\hat{x}\times\hat{y})\frac{\partial E_y}{\partial x} + (\hat{x}\times\hat{z})\frac{\partial E_z}{\partial x}$$

$$+ (\hat{y}\times\hat{x})\frac{\partial E_x}{\partial y} + (\hat{y}\times\hat{y})\frac{\partial E_y}{\partial y} + (\hat{y}\times\hat{z})\frac{\partial E_z}{\partial y}$$

$$+ (\hat{z}\times\hat{x})\frac{\partial E_x}{\partial z} + (\hat{z}\times\hat{y})\frac{\partial E_y}{\partial z} + (\hat{z}\times\hat{z})\frac{\partial E_z}{\partial z}$$



$$(\hat{x}\times\hat{x}) = (\hat{y}\times\hat{y}) = (\hat{z}\times\hat{z}) = 0$$

Right Hand Rule

$$\Rightarrow (\hat{x}\times\hat{y}) = -(\hat{y}\times\hat{x}) = \hat{z}$$

$$(\hat{y}\times\hat{z}) = -(\hat{z}\times\hat{y}) = \hat{x}$$

$$(\hat{z}\times\hat{x}) = -(\hat{x}\times\hat{z}) = \hat{y}$$

Substitute in to get

$$\vec{\nabla} \times \vec{E} = 0 + \hat{z}\frac{\partial E_y}{\partial x} - \hat{y}\frac{\partial E_z}{\partial x}$$

$$- \hat{z}\frac{\partial E_x}{\partial y} + 0 + \hat{x}\frac{\partial E_z}{\partial y}$$

$$+ \hat{y}\frac{\partial E_x}{\partial z} - \hat{x}\frac{\partial E_y}{\partial z} + 0$$

$$= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

(3) Think permutations!

$x\,y\,z \rightarrow y\,z\,x \rightarrow z\,x\,y$ rotate by one each step

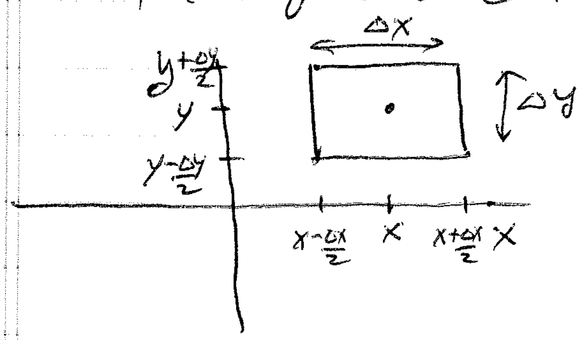
unitvector derivative component of vector of 1st term in each component

$$\vec{\nabla} \times \vec{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$x\,y\,z \longrightarrow y\,z\,x \longrightarrow z\,x\,y$

Physical meaning of curl

Consider an infinitesmal rectangular flat surface. We can always choose coordinates so that the normal to the surface is $\hat{z}$. The surface is centered at position $\vec{r} = (x, y, z)$



surface is in xy plane at height z. Normal is $\hat{n} = \hat{z}$

Consider the line integral of the vector field going counterclockwise around the bounding curve C of the surface S.

$$\oint_C \vec{E} \cdot d\vec{s} = \int_{\text{bottom}} \vec{E} \cdot d\vec{s} + \int_{\text{right}} \vec{E} \cdot d\vec{s} + \int_{\text{top}} \vec{E} \cdot d\vec{s} + \int_{\text{left}} \vec{E} \cdot d\vec{s}$$

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \, E_x(x', y-\tfrac{\Delta y}{2}) + \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \, E_y(x+\tfrac{\Delta x}{2}, y')$$

$d\vec{s} = \hat{x}\,dx$ $\qquad$ $d\vec{s} = \hat{y}\,dy$

$$+ \int_{x+\frac{\Delta x}{2}}^{x-\frac{\Delta x}{2}} dx' \, E_x(x, y+\tfrac{\Delta y}{2}) + \int_{y+\frac{\Delta y}{2}}^{y-\frac{\Delta y}{2}} dy' \, E_y(x-\tfrac{\Delta x}{2}, y')$$

$d\vec{s} = -\hat{x}\,dx$ $\qquad$ $d\vec{s} = -\hat{y}\,dy$

minus sign is handled by the order of the limits in the integral

Consider piece from the ~~top~~ bottom side.

Expand

$$E_x(x', y - \tfrac{\Delta y}{2}) \approx E_x(x', y) + \frac{\partial E_x}{\partial y}(x', y)\left(-\frac{\Delta y}{2}\right)$$

Consider piece from the top side

Expand

$$E_x(x', y + \tfrac{\Delta y}{2}) \approx E_x(x', y) + \frac{\partial E_x}{\partial y}(x', y)\left(\frac{\Delta y}{2}\right)$$

Add integrals over top and bottom pieces

$$\int_{x+\frac{\Delta x}{2}}^{x-\frac{\Delta x}{2}} dx'\, E_x(x', y + \tfrac{\Delta y}{2}) + \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx'\, E_x(x', y - \tfrac{\Delta y}{2})$$

↑ reverse order of limits. This gives a minus sign

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \left[-E_x(x', y + \tfrac{\Delta y}{2}) + E_x(x', y - \tfrac{\Delta y}{2}) \right]$$

insert expansions

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \left[-E_x(x', y) - \frac{\partial E_x(x', y)}{\partial y}\frac{\Delta y}{2} + E_x(x', y) - \frac{\partial E_x(x', y)}{\partial y}\frac{\Delta y}{2} \right]$$

$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \left[-\frac{\partial E_x(x', y)}{\partial y}\Delta y \right]$$

if Δx small, $E_x(x', y)$ is roughly constant over integration interval

$$\simeq -\frac{\partial E_x}{\partial y}(x, y)\, \Delta x \Delta y$$

Similarly one can write the sum of the integrals over the right and left sides

$$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy'\, E_y(x+\tfrac{\Delta x}{2}, y') \;+\; \int_{y+\frac{\Delta y}{2}}^{y-\frac{\Delta y}{2}} dy'\, E_y(x-\tfrac{\Delta x}{2}, y')$$

reverse order of limits to get minus sign

$$= \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \left[E_y(x+\tfrac{\Delta x}{2}, y') - E_y(x-\tfrac{\Delta x}{2}, y') \right]$$

expand for small Δx

$$= \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \left[E_y(x,y') + \frac{\partial E_y}{\partial x}(x,y')\frac{\Delta x}{2} - E_y(x,y') - \frac{\partial E_y}{\partial x}(x,y')\left(-\frac{\Delta x}{2}\right) \right]$$

$$= \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy'\, \frac{\partial E_y}{\partial x}(x,y')\,\Delta x$$

if Δy small, $\frac{\partial E_y}{\partial x}(x,y')$ is roughly constant over integration interval

$$\approx \frac{\partial E_y}{\partial x}(x,y)\,\Delta x\,\Delta y$$

Add all pieces to get

$$\oint_C \vec{E}\cdot d\vec{s} = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)(\Delta x\,\Delta y)$$

z-component of $\vec{\nabla}\times\vec{E}$

can rewrite as:

$$\oint_C \vec{E}\cdot d\vec{s} = \hat{n}\cdot(\vec{\nabla}\times\vec{E})\,\Delta a$$

area of surface

normal to surface

Since we chose coordinates so that $\hat{n} = \hat{z}$, the above actually holds for surfaces with any orientation of $\hat{n}$, since we have written it in a coordinate free vector notation. So

$$\oint_S \vec{E} \cdot d\vec{s} = \hat{n} \cdot (\vec{\nabla} \times \vec{E}) \, \Delta a$$

$$= (\vec{\nabla} \times \vec{E}) \cdot \vec{\Delta a}$$

$\vec{\Delta a}$ is vector area
$\vec{\Delta a} = \hat{n} \, \Delta a$

Although we demonstrated the above only for rectangular planar surfaces, it can be shown (see your math class!) that it holds true for any shape planar surface.

$\Rightarrow$ we can write

$$\boxed{(\vec{\nabla} \times \vec{E}) \cdot \hat{n} = \frac{\oint \vec{E} \cdot d\vec{s}}{\Delta a}}$$

true as $\Delta a \to 0$

the component of $\vec{\nabla} \times \vec{E}$ in the direction $\hat{n}$ at position $\vec{r}$ is the line integral of $\vec{E}$ around a small surface centered at $\vec{r}$, with normal to the surface $\hat{n}$, divided by the area of the surface.

Just as divergence was the flux of the vector field per volume, the curl is the circulation of the vector field per area.