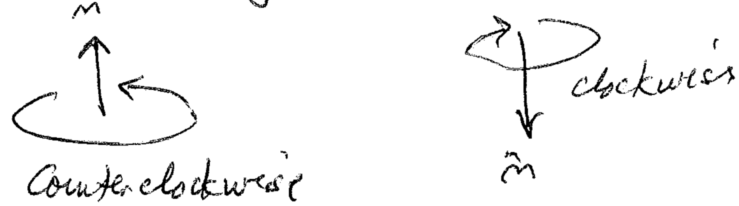
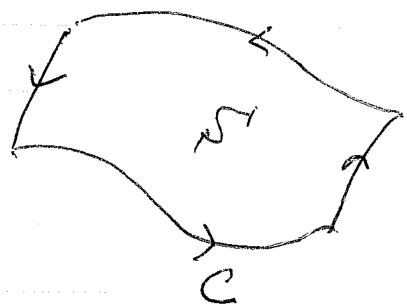
The direction of doing the line integral must be consistent with the direction of the normal $\hat{n}$ according to the Right Hand Rule



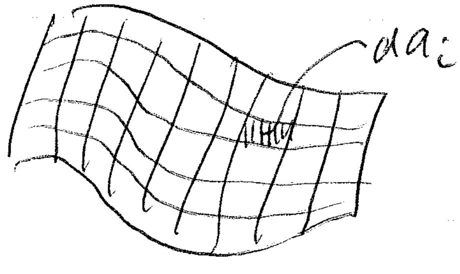
align thumb of <u>right</u> <u>hand</u> along $\hat{n}$, then fingers curl in direction of the line integral

<u>Stokes Theorem of vector calculus</u>

Consider now any surface S bounded by the curve C. The surface need <u>not</u> be planar



We can now tile S with lots of infinitesmally small tiles with area da_i, normals $\hat{n}_i$ and boundary curves C_i

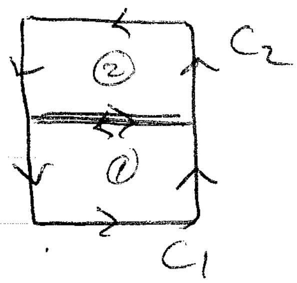


We have

$$\oint_C \vec{E}\cdot d\vec{s} = \sum_i \oint_{C_i} \vec{E}\cdot d\vec{s}$$

This is because all the internal sides of the tiles da_i will cancel pairwise with their neighbor leaving only the terms along the boundary curve of S

For example



$C_{1,2}$ is the boundary of box 1, 2

C is outer boundary of the two boxes

$$\oint_{C_1} \vec{E}\cdot d\vec{s} + \oint_{C_2} \vec{E}\cdot d\vec{s} = \oint_{C} \vec{E}\cdot d\vec{s}$$

because the integrals over the common line segment between box 1 and box 2 ~~cat~~ cancel since they are traveled in opposite directions in C_1 vs C_2.

So $$\oint_C \vec{E}\cdot d\vec{s} = \sum_i \oint_{C_i} \vec{E}\cdot d\vec{s}$$

$$= \sum_i (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da_i = \sum_i (\vec{\nabla}\times\vec{E})\cdot d\vec{a}_i$$

$$= \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} \qquad \text{as } da_i \to 0$$

This is <u>Stokes Theorem</u>!

$$\boxed{\oint_C \vec{E}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a}}$$

line integral of $\vec{E}$ around a closed loop C is equal to the flux of $\vec{E}$ through any surface S bounded by the loop C. Direction of doing line integral must be consistent with direction of normal to S by the right <u>hand rule</u>.

Consider $\vec{E}$ the electric field in an electrostatic problem. We had

$$\oint_C \vec{E}\cdot d\vec{s} = 0 \quad \text{for all loops } C$$

$$\Rightarrow \quad \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = 0 \quad \text{over all surfaces } S$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{E} = 0$$

This completes our two laws of electrostatics

<u>integral form of laws</u>   <u>differential form of laws</u>

Gauss $\quad \oint_S \vec{E}\cdot d\vec{a} = 4\pi Q_{encl} \quad \Leftrightarrow \quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$$\oint_C \vec{E}\cdot d\vec{s} = 0 \quad \Leftrightarrow \quad \vec{\nabla}\times\vec{E} = 0$$

we also have the formulation in terms of the electrostatic potential ϕ

$$\vec{E} = -\vec{\nabla}\phi \quad \Rightarrow \quad -\nabla^2\phi = 4\pi\rho \quad \text{Poisson Eq}$$

$$\vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla}\times\vec{\nabla}\phi = 0$$

$-\oint \vec{A}\cdot d\vec{x}$
$= \oint \vec{\nabla}A\cdot d\vec{s}$
$= 0$

always true: the curl of the gradient of any scalar function is always zero.

Note: $-\int_{r_1}^{r_2} \vec{E}\cdot d\vec{s} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla}\phi\cdot d\vec{s} = \phi(\vec{r}_2)-\phi(\vec{r}_1)$

So $\oint \vec{E}\cdot d\vec{s} = 0$ automatically follows from

$\vec{E} = -\vec{\nabla}\phi$

Since $\oint \vec{E}\cdot d\vec{s} = 0 \Rightarrow \vec{\nabla}\times\vec{E} = 0$

we have in general that $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$
curl of gradient of <u>any</u> scalar function
always vanishes

Units of electrostatic potential

$$\phi = -\int \vec{E} \cdot d\vec{s}$$

in CGS $\vec{E}$ has units of esu/cm^2 $= \dfrac{\text{dyn}}{\text{esu}}$

$$E = \frac{q}{r^2} \qquad = \frac{F}{q}$$

$$\Rightarrow \phi = \frac{\text{dyn cm}}{\text{esu}} = \text{"stat volt"}$$

in MKS $\vec{E}$ has units of $\dfrac{N}{\text{Coul}}$

$$\Rightarrow \phi = \frac{N \cdot m}{\text{coul}} = \text{"volt"}$$

to get conversion factor between volts and statvolts,

$$1 \text{ volt} = \frac{N \cdot m}{\text{coul}} = \frac{10^5 \text{ dyn} \cdot 10^2 \text{ cm}}{\text{coul}} = \frac{10^7 \text{ dyn cm}}{\text{coul}}$$

$$= \frac{10^7 \text{ dyn cm}}{3 \times 10^9 \text{ esu}} = \frac{1}{300} \frac{\text{dyn cm}}{\text{esu}}$$

$$= \frac{1}{300} \text{ statvolt}$$

or $\boxed{300 \text{ volt} = 1 \text{ statvolt}}$

$$U = \frac{1}{2}\int \rho \, \phi \, dV$$

$$= \frac{1}{2}\int \left(\frac{-\nabla^2\phi}{4\pi}\right)\phi \, dV$$

Use $\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = (\vec{\nabla}\phi)^2 + \phi\nabla^2\phi$

$$U = \frac{1}{8\pi}\int dV\left[-\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) + (\vec{\nabla}\phi)^2\right]$$

$$= \frac{1}{8\pi}\int dV \, E^2 - \underbrace{\frac{1}{8\pi}\int d\vec{S}\cdot\phi\vec{\nabla}\phi}$$

If choose $\phi \to 0$ as $r \to \infty$
this integral vanishes

$$U = \frac{1}{8\pi}\int dV E^2$$

$$= \frac{1}{2}\int dV \rho\phi$$

$$\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = \frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\phi\frac{\partial\phi}{\partial z}\right)$$

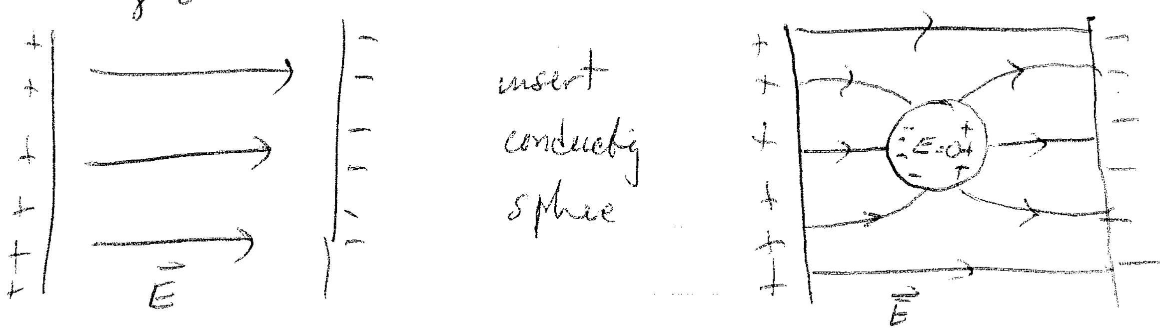
$$= \phi\frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial\phi}{\partial x}\right)^2 + \phi\frac{\partial^2\phi}{\partial y^2} + \left(\frac{\partial\phi}{\partial y}\right)^2 + \phi\frac{\partial^2\phi}{\partial z^2} + \left(\frac{\partial\phi}{\partial z}\right)^2$$

$$= \phi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

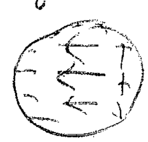
Conductors in electrostatics - conductors contain mobil charges

i) The $\vec{E}$ field inside a conductor must vanish. Otherwise the mobil charges in the conductor would feel a force $q\vec{E}$ and would move - if charges are moving we are not in an electrostatic situation!

Charges move in a way to set up a counter electric field that makes the total electric field zero



-e electrons in sphere move to left side leaving excess of +e charge on right side.



This creates electric field in sphere that exactly cancels out the field from the charged flat planes

Consequences: a) electrostatic potential ϕ = constant inside a conductor. Since $\vec{E} = -\vec{\nabla}\phi$ if ϕ were not constant, then we would have $\vec{E} \neq 0$.

1) Any **net** charge on a conductor must lie on the surface of the conductor. This follows from Gauss law. For any volume inside the conductor bounded by surface S,

$$\oint_S \vec{E} \cdot d\vec{a} = 4\pi Q_{encl}$$

but $\vec{E} = 0$ inside $\Rightarrow Q_{encl} = 0$ so no net charge inside conductor

2) The $\vec{E}$ field at the surface of a conductor must point perpendicular to the surface. If not, there would be a force on charges at the surface moving them along the surface, and again we would not be in an electrostatic situation.

Alternatively, since $\phi = $ const inside conductor, the surface of the conductor is an equipotential (surface of constant ϕ). Since $\vec{E} = -\vec{\nabla}\phi$ and $\vec{\nabla}\phi$ must always point $\perp$ to surfaces of constant ϕ, $\vec{E}$ is $\perp$ to surface.

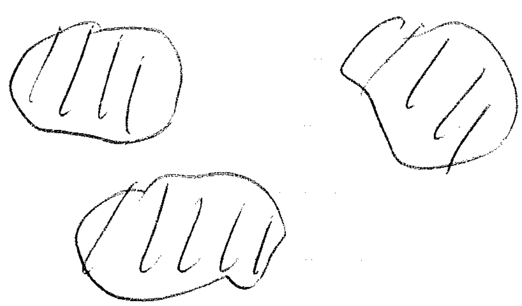
3) At the surface of the conductor $\vec{E} = 4\pi\sigma\hat{n}$ where σ is the surface charge density and $\hat{n}$ is the outward pointing normal. This follows from our general result for a charged surface

$$\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\hat{n}$$

For a conductor, $\vec{E}_{below} = 0$ since $\vec{E} = 0$ inside

Solving problems for $\vec{E}$ in the presence of conductors is not as straightforward as when you are told explicitly the charge distribution $\rho(\vec{r})$. The reason is that with conductors, we do not a priori know where the charge is. The charge on the conductor will reposition itself so as to make $\vec{E}=0$ inside the conductor and $\vec{E} \perp$ surface. We therefore have to somehow self consistently determine both $\vec{E}$ and the location of the charges on the conductor.

The typical problem we have to solve is



a set of conductors ~~on which~~ each of which is specified to be at at constant potential ϕ_i or to have a total charge Q_i. In between conductors, if there is no charge, then $\nabla^2\phi = 0$. Such a problem where ϕ satisfies a differential eqn within a given ~~~~ region, and that must satisfy given conditions on the boundary of that region is called a <u>boundary</u>

value problem. One can show that the above electrostatic problem with conductors always has a unique solution.

Proof for the case where each conductor i is specified to be at potential ϕ_i, and $\phi \to 0$ as $\vec{r} \to \infty$

Suppose we had two solutions ϕ and ϕ', ie $\nabla^2 \phi = 0$ and $\nabla^2 \phi' = 0$ between the conductors while $\phi(\vec{r}) = \phi_i$ and $\phi'(\vec{r}) = \phi_i$ for $\vec{r}$ on surface conductor i.

Then consider $W = \phi - \phi'$. Clearly $\nabla^2 W = 0$ between conductors and $W(\vec{r}) = 0$ for $\vec{r}$ on surface conductor i, and $W \to 0$ as $\vec{r} \to \infty$

Then we must have $W = 0$ everywhere otherwise W would have either a local maximum or minimum somewhere in space between the conductors. But $\nabla^2 W = 0$ there and harmonic functions can have no local max or min. Hence $W = 0$ so $\phi = \phi'$ and there is only one unique solution