The direction of doing the line integral must be consistent with the direction of the normal \( \hat{\mathbf{n}} \) according to the Right Hand Rule. Align the thumb of right hand along \( \hat{\mathbf{n}} \), then fingers curl in direction of the line integral.

**Stokes Theorem of Vector Calculus**

Consider now any surface \( S \) bounded by the curve \( C \). The surface need not be planar.

We can now tile \( S \) with lots of infinitesimally small tiles with area \( da_i \), normal \( \hat{\mathbf{n}}_i \), and boundary curves \( C_i \).

We have

\[
\int_S \mathbf{E} \cdot d\mathbf{A} = \sum_i \int_{C_i} \mathbf{E} \cdot d\mathbf{s}
\]

This is because all the internal sides of the tiles \( da_i \) will cancel pairwise with their neighbors leaving only the terms along the boundary curve of \( S \).
For example,

\[ \oint_{C_1} \mathbf{E} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{E} \cdot d\mathbf{s} = \oint_{C} \mathbf{E} \cdot d\mathbf{s} \]

\[ C_1, C_2 \text{ is the boundary of box 1, 2} \]

C is outer boundary of the two boxes

Because the integrals over the common line segment between box 1 and box 2 cancel since they are traveled in opposite directions in \( C_1 \) vs \( C_2 \).

So

\[ \oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = \sum_{\mathbf{c}_1} \oint_{C_1} \mathbf{E} \cdot d\mathbf{s} \]

\[ = \sum_{\mathbf{c}_1} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \, d\mathbf{a}_1 = \sum_{\mathbf{c}_1} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \, d\mathbf{a}_1 \]

\[ = \int_{S} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \, d\mathbf{a} \quad \text{as } d\mathbf{a} \to 0 \]

This is Stokes’ Theorem!

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{s} = \int_{S} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \, d\mathbf{a} \]

The line integral of \( \mathbf{E} \) around a closed loop \( C \) is equal to the flux of \( \mathbf{E} \) through any surface \( S \) bounded by the loop \( C \). Direction of doing line integral must be consistent with direction of normal to \( S \) by the right-hand rule.
Consider $\vec{E}$ the electric field in an electrostatic problem. We had

$$\oint \vec{E} \cdot d\vec{s} = 0 \quad \text{for all loops } C$$

$$\Rightarrow \int_{\Sigma} (\nabla \times \vec{E}) \cdot d\vec{a} = 0 \quad \text{over all surfaces } \Sigma$$

$$\Rightarrow \nabla \times \vec{E} = 0$$

This completes our two laws of electrostatics:

<table>
<thead>
<tr>
<th>Integral form of laws</th>
<th>Differential form of laws</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\oint \vec{E} \cdot d\vec{s} = 4\pi Q_{\text{enc}}$</td>
<td>$\nabla \cdot \vec{E} = 4\pi \rho$</td>
</tr>
<tr>
<td>$\oint \vec{E} \cdot d\vec{s} = 0$</td>
<td>$\nabla \times \vec{E} = 0$</td>
</tr>
</tbody>
</table>

we also have the formulation in terms of the electrostatic potential $\phi$:

$$\vec{E} = -\nabla \phi \quad \Rightarrow \quad -\nabla^2 \phi = 4\pi \rho \quad \text{Poisson Eq}$$

$$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \nabla \times \nabla \phi = 0$$

always true: the curl of the gradient of any scalar function is always zero.
Note: \[ -\int_{\vec{n}_1} \vec{E} \cdot d\vec{s} = \int_{\vec{n}_2} \vec{E} \cdot d\vec{s} = \Phi(\vec{r}_2) - \Phi(\vec{r}_1) \]

So \[ \int \vec{E} \cdot d\vec{s} = 0 \] automatically follows from \[ \vec{E} = -\vec{\nabla} \Phi \]

Since \[ \int \vec{E} \cdot d\vec{s} = 0 \] \( \Rightarrow \) \[ \vec{\nabla} \times \vec{E} = 0 \]

we have in general that \[ \vec{\nabla} \times (\vec{\nabla} \Phi) = 0 \] curl of gradient of any scalar function always vanishes
Units of electrostatic potential

\[ \phi = - \int E \cdot ds \]

in CGS, \( E \) has units of \( \text{esu/cm}^2 \) = \( \text{dyn} \)

\[ E = \frac{q}{r^2} = \frac{F}{q} \]

\[ \Rightarrow \phi = \frac{\text{dyn cm}}{\text{esu}} = \text{"stat volt"} \]

in MKS, \( E \) has units of \( \frac{N}{\text{Coul}} \)

\[ \Rightarrow \phi = \frac{N \cdot m}{\text{Coul}} = \text{"volt"} \]

To set conversion factor between volts and statvolts,

1 volt = \( \frac{N \cdot m}{\text{Coul}} = \frac{10^5 \text{ dyn} \cdot 10^2 \text{ cm}}{\text{Coul}} = \frac{10^7 \text{ dyn cm}}{\text{Coul}} \]

= \( \frac{10^7 \text{ dyn cm}}{3 \times 10^9 \text{ esu}} = \frac{1}{300} \frac{\text{dyn cm}}{\text{esu}} \)

\[ = \frac{1}{300} \text{ stat volt} \]

or \[ 300 \text{ volt} = 1 \text{ stat volt} \]
\[ U = \frac{1}{2} \int p \phi \, dV \]
\[ = \frac{1}{2} \int \left( -\nabla^2 \phi \right) \phi \, dV \]
\[ \text{where} \quad \nabla^2 (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi \]

\[ U = \frac{1}{8\pi} \int dV \left[ -\nabla \cdot (\phi \nabla \phi) + (\nabla \phi)^2 \right] \]
\[ = \frac{1}{8\pi} \int dV \, E^2 \quad - \quad \frac{1}{8\pi} \int dS \cdot \nabla \phi \]

If \( \phi \to 0 \) as \( r \to \infty \), the integral vanishes.

\[ U = \frac{1}{8\pi} \int dV \, E^2 \]
\[ = \frac{1}{2} \int \nabla p \, \phi \]

\[ \nabla \cdot (\phi \nabla \phi) = \frac{\partial}{\partial x} \left( \phi \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \phi \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \phi \frac{\partial \phi}{\partial z} \right) \]
\[ = \phi \frac{\partial^2 \phi}{\partial x^2} + \left( \frac{\partial \phi}{\partial x} \right)^2 + \phi \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{\partial \phi}{\partial y} \right)^2 + \phi \frac{\partial^2 \phi}{\partial z^2} + \left( \frac{\partial \phi}{\partial z} \right)^2 \]
\[ = \phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi \]
Conductors in electrostatics - conductors contain mobile charges.

The \( \mathbf{E} \) field inside a conductor must vanish. Otherwise, the mobile charges in the conductor would feel a force \( q \mathbf{E} \) and would move - if charges are moving we are not in an electrostatic situation!

Charges move in a way to set up a counter-electric field that makes the total electric field zero.

- Electrons in sphere move to left side leaving excess of + charge on right side.

This creates electric field in sphere that exactly cancels out the field from the charged flat planes.

Consequences:

1. Electrostatic potential \( \Phi \) is constant inside a conductor. Since \( \mathbf{E} = -\nabla \Phi \), if \( \Phi \) were not constant, the we would have \( \mathbf{E} \neq 0 \).
b) Any net charge on a conductor must lie on the surface of the conductor. This follows from Gauss's law. For any volume inside the conductor bounded by surface \( S \),

\[
\iint_S \vec{E} \cdot d\vec{a} = 4\pi Q_{\text{enclosed}}
\]

but \( \vec{E} = 0 \) inside \( \Rightarrow Q_{\text{enclosed}} = 0 \) so no net charge inside conductor.

2) The \( \vec{E} \) field at the surface of a conductor must point perpendicular to the surface. If not, there would be a force on charges at the surface moving them along the surface, and again we would not be in an electrostatic situation.

Alternatively, since \( \Phi = \text{const} \) inside conductor, the surface of the conductor is an \underline{equipotential} (surface of constant \( \Phi \)). Since \( \vec{E} = -\nabla \Phi \) and \( \nabla \Phi \) must always point \( \perp \) to surfaces of constant \( \Phi \), \( \vec{E} \) is \( \perp \) to surface.

3) At the surface of the conductor \( \vec{E} = 4\pi Q \hat{n} \)
where \( Q \) is the surface charge density and \( \hat{n} \) is the outward pointing normal. This follows from our general result for a charged surface

\[
\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = 4\pi Q \hat{n}
\]

For a conductor, \( \vec{E}_{\text{below}} = 0 \) since \( \vec{E} = 0 \) inside.
Solving problems for \( \vec{E} \) in the presence of conductors is not as straightforward as when you are told explicitly the charge distribution \( \rho \). The reason is that with conductors, we do not a priori know where the charge is. The charge on the conductor will reposition itself so as to make \( \vec{E} = 0 \) inside the conductor and \( \vec{E} \neq 0 \) surface. We therefore have to somehow self-consistently determine both \( \vec{E} \) and the location of the charges on the conductor.

The typical problem we have to solve is

\[ a \text{ set of conductors on which each of which is specified to be at constant potential } \phi_i \text{ or to have a total charge } Q_i. \text{ In between conductors, if there is no charge, then } \nabla^2 \phi = 0. \text{ Such a problem where } \phi \text{ satisfies differential equation with a given region, and that must satisfy given conditions on the boundary of that region is called a boundary } \]
value problem. One can show that the above electrostatic problem with conductors always has a unique solution.

Proof for the case where each conductor $\mathcal{C}_i$ is specified to be at potential $\phi_i$ as $r \to \infty$.

Suppose we had two solutions $\phi$ and $\phi'$, i.e.
$\nabla^2 \phi = 0$ and $\nabla^2 \phi' = 0$ between the conductors while $\phi(\mathcal{C}_i) = \phi_i$ and $\phi'(\mathcal{C}_i) = \phi_i$ for $\mathcal{C}_i$ on surface conductor $\mathcal{C}_i$.

Then consider $W = \phi - \phi'$. Clearly $\nabla^2 W = 0$ between conductors and $W(\mathcal{C}_i) = 0$ for $\mathcal{C}_i$ on surface conductor $\mathcal{C}_i$ and $W > 0$ as $r \to \infty$.

Then we must have $W = 0$ everywhere otherwise $W$ would have either a local max or min somewhere in space between the conductors. But $\nabla^2 W = 0$ there as harmonic functions can have no local max or min. Hence $W = 0$ so $\phi = \phi'$ and there is only one unique solution.