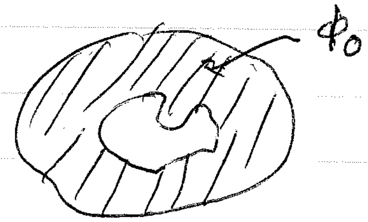
Consider a charge free cavity with a conductor




Since the conductor is at a constant potential $\phi_0$, and since there are no charges in the cavity $\Rightarrow \nabla^2\phi = 0$ inside cavity

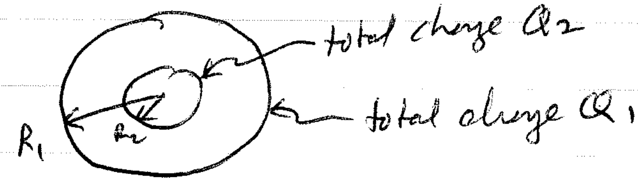
$\Rightarrow$ $\phi = \phi_0$ everywhere inside cavity is one solution to this boundary value problem for $\phi(\vec{r})$ inside the cavity.

By preceding result, $\phi = \phi_0$ inside cavity is thus the only solution

$\Rightarrow$ inside the cavity $\vec{E} = -\vec{\nabla}\phi = -\vec{\nabla}\phi_0 = 0$

$\vec{E} = 0$ inside a charge free cavity inside a conductor.

Example Concentric shells




what is $\phi$ at surfaces $r = R_1$ and $r = R_2$ ?
By spherical symmetry, $Q_1$ and $Q_2$ distributed uniformly on surfaces
Easy to find $\vec{E}(\vec{r}) = E(r)\hat{r}$

$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E(r) = 4\pi Q_{encl} = 4\pi \begin{cases} 0 & r < R_2 \\ Q_2 & R_2 < r < R_1 \\ Q_1 + Q_2 & R_1 < r \end{cases}$$

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < R_2 \\ \frac{Q_2}{r^2}\hat{r} & R_2 < r < R_1 \\ \frac{Q_1+Q_2}{r^2}\hat{r} & R_1 < r \end{cases}$$

Potential $\quad \phi(r) = -\int_\infty^r dr'\, E(r')$ $\quad$ reference pt at $\infty$ radial path

$r > R_1$

$$\phi(r) = -\int_\infty^r dr' \left(\frac{Q_1+Q_2}{r'^2}\right) = \frac{Q_1+Q_2}{r}$$

so $\phi(R_1) = \dfrac{Q_1+Q_2}{R_1}$

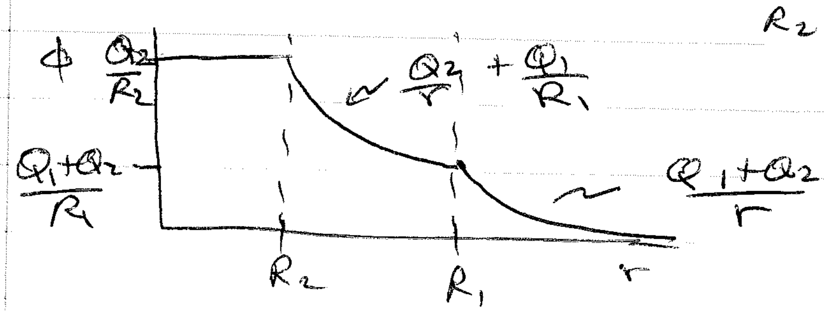
$R_1 > r > R_2$ $\quad \phi(r) = -\int_\infty^r dr'\, E(r') = -\int_\infty^{R_1} dr'\, E(r') - \int_{R_1}^r dr'\, E(r')$

$$= \phi(R_1) - \int_{R_1}^r dr' \frac{Q_2}{r'^2} = \phi(R_1) + \frac{Q_2}{r} - \frac{Q_2}{R_1}$$

$$= \frac{Q_1+Q_2}{R_1} + \frac{Q_2}{r} - \frac{Q_2}{R_1} = \frac{Q_1}{R_1} + \frac{Q_2}{r}$$

$$\phi(R_2) = \frac{Q_1}{R_1} + \frac{Q_2}{R_2}$$

$r < R_2$ $\quad \phi(r) = \phi(R_2) - \int_{R_2}^{r} dr'\, E(r') = \phi(R_2)$




If $Q_1 = -Q_2$ then $\phi = 0 \quad r > R_1$

$\Rightarrow \vec{E} = 0 \quad r > R_1$

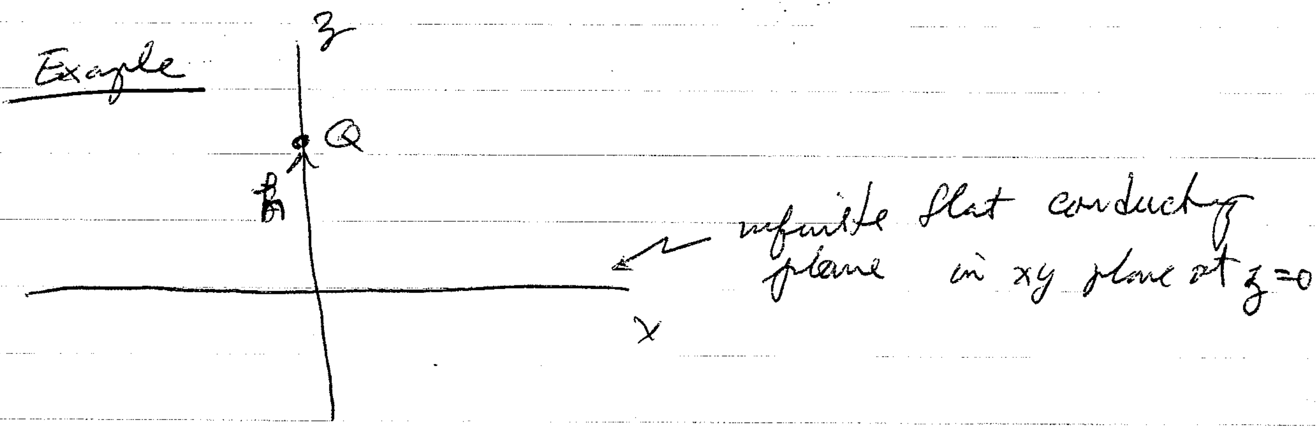
then $\vec{E} \neq 0$ only for $R_2 < r < R_1$ inbetween shells

Another way to solve: superposition

from shell 1 $\quad \phi_1(r) = \begin{cases} \frac{Q_1}{r} & r > R_1 \\ \frac{Q_1}{R_1} & r \leq R_1 \end{cases}$ $\qquad$ from shell 2 $\quad \phi_2(r) = \begin{cases} \frac{Q_2}{r} & r > R_2 \\ \frac{Q_2}{R_2} & r \leq R_2 \end{cases}$

$$\phi = \phi_1 + \phi_2 = \begin{cases} \frac{Q_2}{R_2} & r < R_2 \\ \frac{Q_1}{R_1} + \frac{Q_2}{r} & R_2 < r < R_1 \\ \frac{Q_1}{r} + \frac{Q_2}{r} & R_1 < r \end{cases}$$

## Example

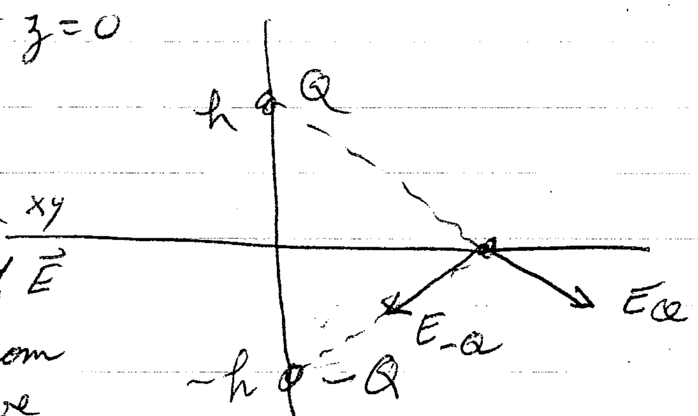



① Suppose conducting plane is grounded, ie
$\phi(x,y,0) = 0$

We know the electric field must be $\perp$ to the conducting plane.

Method of image charges – a nice trick.

Put $-Q$ a distance ~~h~~ below the plane. The electric field from $+Q$ above the plane and $-Q$ below the plane has all the desired properties of our solution: Near $Q$ it looks just like a pt charge, while $\vec{E}$ $\perp$ to xy plane at $z=0$




we see that at each point on the xy plane, the ~~sum of~~ $\vec{E}$ from Q and $\vec{E}$ from image $-Q$ have components in xy plane that cancel. Sum is purely in $\hat{z}$-direction

Since this trick gives an $\vec{E}$ field with the required properties, let see what potential $\phi$ it gives. For $Q$ at $h\hat{z}$ and $-Q$ at $-h\hat{z}$

$$\phi(\vec{r}) = \frac{Q}{|\vec{r}-h\hat{z}|} + \frac{-Q}{|\vec{r}+h\hat{z}|}$$

potential at $\vec{r}$ from $+Q$ at $\vec{r}' = h\hat{z}$

potential at $\vec{r}$ from $-Q$ at $\vec{r}' = -h\hat{z}$

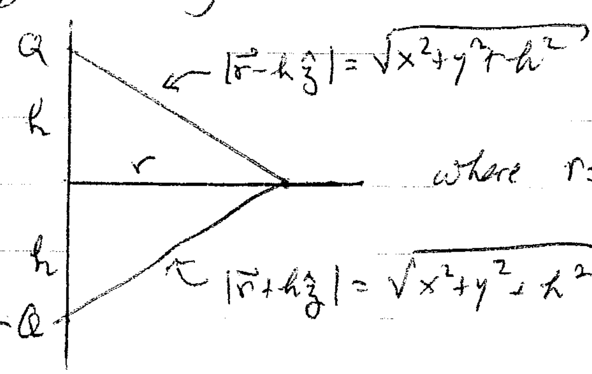
denominator is $|\vec{r}-\vec{r}'|$

For $\vec{r} = (x, y, 0)$ on the surface of the conductor we have

$$\vec{r}-h\hat{z} = (x, y, -h) \qquad \vec{r}+h\hat{z} = (x, y, h)$$

$$|\vec{r}-h\hat{z}| = \sqrt{x^2+y^2+h^2} \qquad |\vec{r}-h\hat{z}| = \sqrt{x^2+y^2+h^2}$$

or geometrically:




where $r = \sqrt{x^2+y^2}$

So $$\phi(x, y, 0) = \frac{Q}{\sqrt{x^2+y^2+h^2}} - \frac{Q}{\sqrt{x^2+y^2+h^2}}$$

$$= 0$$

satisfies boundary condition $\phi = 0$ on grounded surface of conductor

Thus we have found a solution to our problem. Since the solution is unique, we know it is the only solution!

We now want to find the charge induced on the surface of the conductor. We can use

$\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma\hat{n}$ for any conducting surface

Here $\hat{n} = \hat{z}$ outward normal
$\vec{E}_{below} = 0$ since $\vec{E} = 0$ inside conductor

$\Rightarrow \vec{E}_{above} = 4\pi\sigma\hat{z}$ or $\sigma(x,y) = \frac{1}{4\pi} E_z(x,y,0)$

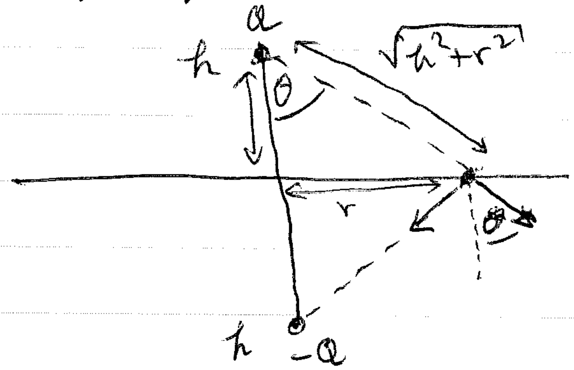
$z$ component of electric field at surface

To get $\sigma$ we therefore need to compute $E_z$. We could in principle get $E_z$ by taking the gradient of $\phi$

$$E_z = -\frac{\partial\phi}{\partial z}$$ since we already solved for $\phi$ on the previous page.

But it is easier just to compute $E_z$ directly from $Q$ and the image $-Q$.

On surface of conductor we have




magnitude of $\vec{E}$ from Q at pt on surface conductor a dist $r=\sqrt{x^2+y^2}$ from origin is

$$|\vec{E}| = \frac{Q}{h^2+r^2}$$

projection in z-direction is $E_z = \frac{-Q}{h^2+r^2}\cos\theta$

$$= \frac{-Q}{h^2+r^2}\frac{h}{\sqrt{h^2+r^2}}$$

when add in field from $-Q$ we double above
So at surface of conductor

$$\vec{E}(x,y,0) = \frac{-2Qh}{(h^2+x^2+y^2)^{3/2}}\hat{z}$$

$$= \frac{-2Qh}{(h^2+r^2)^{3/2}}\hat{z}$$

We can now compute the surface charge density on the surface of the conducting plane using

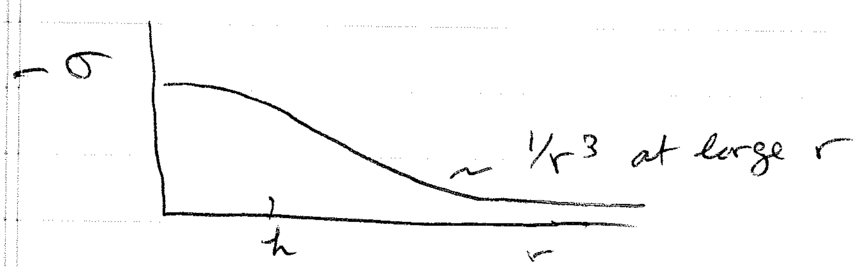
$$\vec{E} = 4\pi\sigma\,\hat{n} \quad \text{on surface conductor}$$

here $\hat{n} = \hat{z}$ outward normal

so

$$\vec{E}\cdot\hat{z} = 4\pi\sigma \Rightarrow \sigma(r) = -\frac{2Qh}{4\pi(h^2+r^2)^{3/2}}$$

largest at $r=0$, goes to zero as $r\to\infty$




What is the total charge on the surface of the conductor?

$$2\pi\int_0^\infty dr\, r\,\sigma(r) = -\frac{4\pi Qh}{4\pi}\int_0^\infty dr\,\frac{r}{(h^2+r^2)^{3/2}}$$

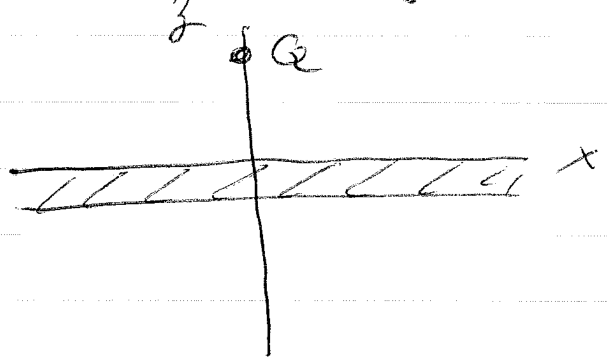
$$= -\cancel{4\pi}\,Qh\left[\frac{-1}{(h^2+r^2)^{1/2}}\right]_0^\infty$$

$$= -\frac{\cancel{4\pi}\,Qh}{h} = -Q$$

total charge is just $-Q$ ! same amount of charge as image charge.

The preceeding problem was for a grounded conductor with $\phi = 0$ on surface. The $+Q$ in front of the conducting surface induced a total charge $-Q$ to flow onto the surface of the conductor.

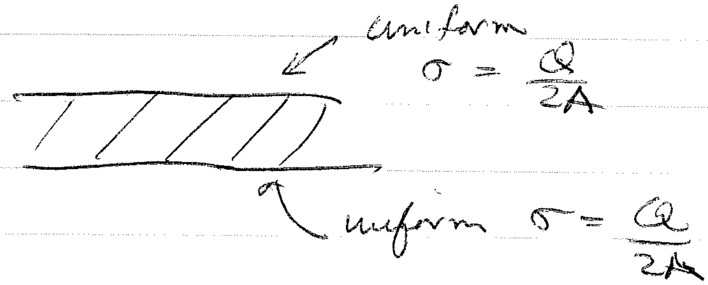
What if we had a neutral conducting slab of finite width? Then the total charge on the slab must always sum to zero.




We can make use of our previous solution

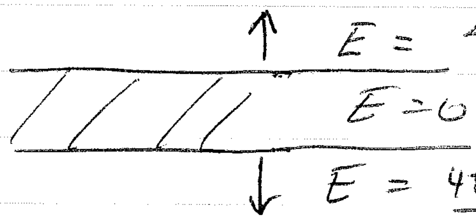
We can put the same $\sigma(x,y)$ on the upper surface of the conductor as we had in the previous problem. This will give an $\vec{E}$ that is normal to the surface as needed, but it also induces a net charge $-Q$ ~~on~~ the conductor that would violate the constraint that the conductor is neutral. To fix that we have to add a $+Q$ to the conductor in such a way that it does not mess up any of the required properties of our solution, ie $\vec{E}$ normal to surface and $\vec{E} = 0$ inside conducting slab.

The way to do this is to distribute $+\frac{Q}{2}$ uniformly over both the upper and lower surfaces of the conducting slab




$A$ = area of surface

the electric field from this uniform $\sigma$ on both surfaces is

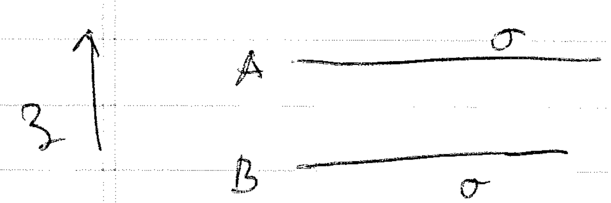


$\uparrow$ $E = \frac{4\pi Q}{2A}$ up

$E = 0$ inside

$\downarrow$ $E = \frac{4\pi Q}{2A}$ down

We get this by superposition:




$$\vec{E}_A = \begin{cases} 2\pi\sigma\,\hat{z} & \text{above A} \\ -2\pi\sigma\,\hat{z} & \text{below A} \end{cases}$$

$$\vec{E}_B = \begin{cases} 2\pi\sigma\,\hat{z} & \text{above A} \\ -2\pi\sigma\,\hat{z} & \text{below A} \end{cases}$$

$$\Rightarrow \vec{E} = \vec{E}_A + \vec{E}_B = \begin{cases} 4\pi\sigma\,\hat{z} & \text{above A} \\ -4\pi\sigma\,\hat{z} & \text{below B} \\ 0 & \text{between A and B} \end{cases}$$

So this gives $E$ that is normal to surface and zero inside slab.

So the total $\vec{E}$ for our problem of the neutral slab is.

for $z>0$ above slab $\vec{E}$ = electric field from $+Q$ at $h\hat{z}$
+ electric field from image $-Q$ at $-h\hat{z}$
+ $\frac{4\pi Q}{2A}\hat{z}$ from $\frac{Q}{2}$ on upper surface

for $z$ below slab $\vec{E} = -\frac{4\pi Q}{2A}\hat{z}$ from $\frac{Q}{2}$ on lower surface

inside slab $\vec{E} = 0$