Capacitance & Capacitors

An isolated conductor with charge Q has a ~~point~~ potential ϕ_0 with respect to infinity.

For example: a conducting sphere of radius a charge Q has:

Electric field $\vec{E}(\vec{r}) = \begin{cases} \frac{Q}{r^2}\hat{r} & r > a \\ 0 & r < a \end{cases}$

potential $\phi(\vec{r}) = \begin{cases} \frac{Q}{r} & r > a \\ \frac{Q}{a} & r < a \end{cases}$

So the sphere is at potential $\phi_0 = \phi(a) = \frac{Q}{a}$

We see that $\phi_0 \propto Q$. This is in general the case no matter what the shape of the conductor.

One defines the capacitance of the conductor by

$$\boxed{C = \frac{Q}{\phi_0}}$$

~~capacitance of conductor with net charge Q~~

Since $Q \propto \phi_0$, C is independent of the total charge on the conductor. It is determined only by the shape of the conductor

units of $C = \frac{\text{esu}}{\text{statvolt}}$ $\quad$ statvolt $= \frac{\text{esu}}{\text{cm}}$

$\Rightarrow C = \frac{\text{esu}}{\text{esu/cm}} = \text{cm}$

in CGS units, the units of capacitance are <u>cm</u>

Since C has units of length, in general C scales with the ~~size~~ length of the conductor

For the sphere $C = \frac{Q}{\Phi_0} = \frac{Q}{(Q/a)} = a$ radius

In MKS units, $C = \frac{Q}{\Phi_0} = \frac{\text{coul}}{\text{volt}}$ = "farad" denoted "F" named after Michael Faraday

$$1 \text{ farad} = 9 \times 10^{11} \text{ cm}$$

need to convert coul → esu
volt → statvolt

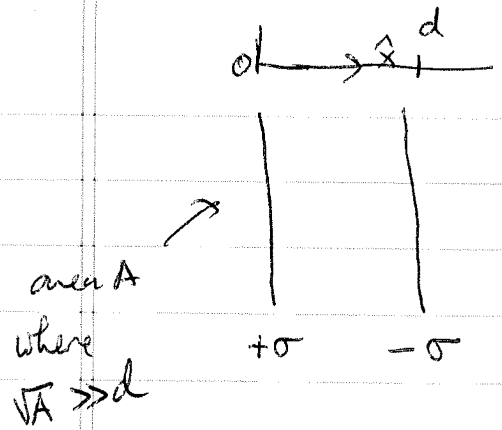
$$\frac{\text{coul}}{\text{volt}} = \frac{3 \times 10^9 \text{ esu}}{\frac{1}{300} \text{ statvolt}} = 900 \times 10^9 \frac{\text{esu}}{\text{statvolt}} = 9 \times 10^{11} \text{ cm}$$

When we have two conducting surfaces, one charged with $+Q$, the other charged with $-Q$, then the capacitance of the two conductors is defined as

$$C = \frac{Q}{\Phi_1 - \Phi_2}$$

where Φ_1 is potential of conductor with $+Q$, and Φ_2 is potential of conductor with $-Q$.

The combination of the two conductors plus any insulating material that is between them is called a <u>capacitor</u>



$$\vec{E} = 0 \qquad x < 0,\ x > d$$

$$\vec{E} = 4\pi\sigma \hat{x} \qquad 0 < x < d$$

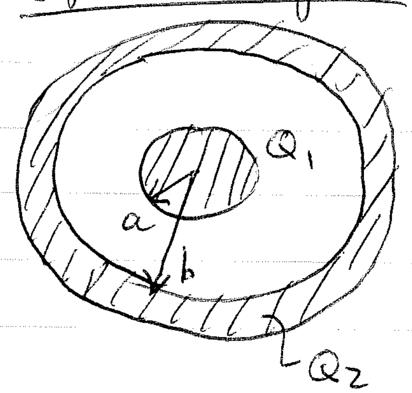
potential $\phi(d) - \phi(0) = -\int_0^d \vec{E} \cdot d\vec{x}$

$$= -4\pi\sigma d$$

Capacitance is

$$C = \frac{\sigma A}{\phi(0) - \phi(d)} = \frac{\sigma A}{4\pi\sigma d} = \frac{A}{4\pi d}$$

<u>spherical capacitor</u>



conductor 1 : sphere radius a
conductor 2 : spherical shell of inner radius b, outer radius c.
Q_1 on conductor 1
Q_2 on conductor 2

The charge on the inner surface of conductor 2 must be $-Q_1$. This is because $\vec{E} = 0$ inside conductor 2 $\Rightarrow$ charge enclosed by a surface of radius $b < r < c$ must contain zero charge.
Charge on outside surface of conductor 2 must therefore be $Q_2 + Q_1$. (If $Q_2 = -Q_1$ then charge on outer wall is zero).

The capacitance of this configuration is defined with respect to conductor 1 and inner surface of conductor 2.

$$C = \frac{Q_1}{\phi(a) - \phi(b)}$$

To get ϕ we first solve for $\vec{E}(\vec{r})$ $a<r<b$

Gauss Law $\oint_{\text{surface radii } r} \vec{E} \cdot d\vec{a} = 4\pi r^2 E(r) = 4\pi Q_1$

$$\vec{E}(\vec{r}) = \frac{Q_1}{r^2}\hat{r}$$

$$\Rightarrow \phi(a) - \phi(b) = -\int_b^a dr\, E(r)$$

$$= \int_a^b dr\, E(r)$$

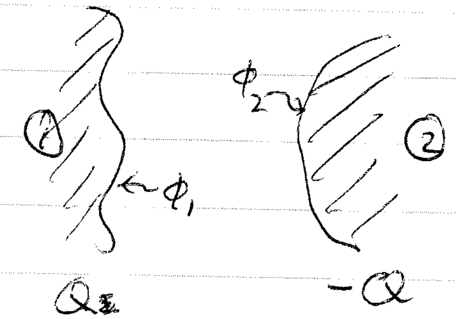
$$= \int_a^b dr\, \frac{Q_1}{r^2}$$

$$= Q_1\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= Q_1\left(\frac{b-a}{ab}\right)$$

$$C = \frac{Q_1}{Q_1\left(\frac{b-a}{ab}\right)} = \frac{ab}{b-a}$$

Energy stored in a capacitor



suppose we take a charge dQ from conductor ② and move it to conductor ①. The work done to move the charge is

$$dW = dQ(\phi_1 - \phi_2)$$

but $\phi_1 - \phi_2 = \dfrac{Q}{C}$

$$dW = \frac{dQ\,Q}{C}$$ where C is indep of the charge on capacitor Q.

What is the work done, starting from an uncharged capacitor ($Q=0$) to charge it up to charge Q?

$$W = \int dW = \int_0^Q \frac{dQ'\,Q'}{C} = \frac{1}{2}\frac{Q^2}{C}$$

This is the electrostatic energy stored in the capacitor

$$W = \frac{1}{2}\frac{Q^2}{C}$$ Since $Q = C\,\phi_{12}$

$\phi_{12} = \phi_1 - \phi_2$

$$W = \frac{1}{2}\frac{(C\phi_{12})^2}{C} = \frac{1}{2}C\phi_{12}^2$$

potential difference from one conductor to the other

For a parallel plate capacitor $C = \frac{A}{4\pi d}$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(\sigma A)^2}{A} 4\pi d = 2\pi\sigma^2 A d$$

Compare to

$$U = \frac{1}{8\pi}\int dV\, |\vec{E}|^2 \qquad \text{between plates } |\vec{E}| = 4\pi\sigma$$

$$= \frac{1}{8\pi}(4\pi\sigma)^2 \cdot Ad = 2\pi\sigma^2 A d$$

or

$$U = \frac{1}{2}\int dV\, \rho\phi = \frac{1}{2}\sigma A(\Phi_1 - \Phi_2)$$

$$\Phi_1 - \Phi_2 = -\int_2^1 \vec{E}\cdot d\vec{s} = \int_1^2 E \cdot ds = \cancel{2\pi\sigma^2}\ 4\pi\sigma d$$

$$U = \frac{1}{2}\sigma A(4\pi\sigma d) = 2\pi\sigma^2 A d.$$

Same result holds for an isolated conductor.
For a conducting sphere of radius a, $C = a$

$$\Rightarrow U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{a} \qquad \text{energy stored in a charged conducting sphere}$$

Compare to $U = \frac{1}{2}\int dV\, \rho\phi = \frac{1}{2}\, 4\pi a^2\sigma\, \phi(a)$

$$= 2\pi a^2\sigma\left(\frac{Q}{a}\right) = 2\pi a^2\sigma\,\frac{Q}{a}$$

$$= \frac{1}{2}(4\pi a^2\sigma)\frac{Q}{a} = \frac{1}{2}\frac{Q^2}{a}$$

Relation between electrostatic potential and potential energy

From mechanics:

The work a force $\vec{F}$ does in moving a particle from $\vec{r}_1$ to $\vec{r}_2$ is

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{s}$$

line integral along path from $\vec{r}_1$ to $\vec{r}_2$

When this integral is independent of the path taken we say that $\vec{F}$ is a conservative force and it has a corresponding potential energy $U(\vec{r})$. U is defined so that the work we do on the particle to move it from $\vec{r}_1$ to $\vec{r}_2$ in the presence of the force $\vec{F}$ is

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{s}$$

change in potential energy

work we do

The $-\vec{F}$ is because the ~~work~~ force we ~~do~~ apply must oppose ~~the~~ and balance out the force $\vec{F}$.

From our discussion of the gradient we can then write

$$\vec{F} = -\vec{\nabla} U$$

Then $-\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{s} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} U \cdot d\vec{s} = U(\vec{r}_2) - U(\vec{r}_1)$

line integral of a gradient is always independent of the path

For the electrostatic electric field $\vec{E}$ we have

$\vec{E} = \frac{\vec{F}}{q}$ and $-\int_{\vec{r}_1}^{\vec{r}_2} q\vec{E} \cdot d\vec{s}$ is independent of the path

$\Rightarrow$ electrostatic force is conservative

$\Rightarrow$ there exists potential energy U

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} q\vec{E} \cdot d\vec{s}$$

or $$\frac{U}{q}(\vec{r}_2) - \frac{U}{q}(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s}$$

$\frac{U}{q}$ is therefore just the electrostatic potential ϕ

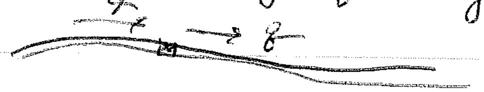
$\phi = \frac{U}{q}$ and just like $\vec{F} = -\vec{\nabla} U$
we have $\vec{E} = -\vec{\nabla}\phi$

ϕ is the potential energy per unit charge of a charged particle in an electrostatic field due to other charges.

Electric Currents

An electric current is due to charges in motion
Units of current is charge/time
in CGS it is esu/s. In MKS it is coul/sec = "amp"

For current flowing down a wire, the current is
steady if the same amount of charge flows past
each point in the wire in the same amount of
time. Every charge that flows out of a
particular segment of the wire is replaced by
another charge flowing into that segment of the wire.



For a wire with a steady current, the
moving charges have a constant average
speed that does not change with time.

Consider a more general situation of
charges moving in space. The current I
flowing through some small area $\Delta\vec{a}$ is the
flux of charge through $\Delta\vec{a}$

$$I = n q \vec{v} \cdot \Delta\vec{a}$$

density of charges — charge per particle — average ~~sp~~ velocity of particles

See notes when we introduced concept of flux.

For many different types of particles i
particle of type i has charge q_i and velocity $\vec{v}_i$
and density n_i

$$I = \sum_i n_i q_i \vec{v}_i \cdot \Delta \vec{a}$$

If just as many $+q$ as $-q$ pass through $\Delta\vec{a}$ in same time, there is no net current flowing
(Ex - water flowing in pipe has no electric current. just as many electrons with $-e$ pass through as do protons with $+e$ - so no net charge flowing)

Look at one type of charged particle, say electrons. If these have different velocities $\vec{v}_k$, then

$$I = \sum_k (-e) n_k \vec{v}_k \cdot \Delta \vec{a}$$

average over all electron velocities.

$$I = -e n \langle \vec{v} \rangle \cdot \Delta \vec{a}$$

density of all electrons

i) For current of negatively charged particles, current flows in <u>opposite</u> direction to the particles average velocity