2) $\langle \vec{v} \rangle = \frac{1}{n} \sum_{k} n_k \vec{v}_k$

It is average velocity of charges that determines the current. In general, the electrons in a wire are traveling in random directions with speeds that can be much greater than $|\langle \vec{v} \rangle|$. It is only the average velocity that gives rise to a net transport of charge.

Thus for several different types of charged particles $i$ we have

average velocity of particles of type $i$

$$I = \sum_i q_i n_i \langle \vec{v}_i \rangle \cdot \Delta \vec{a}$$

charge on particle type $i$

density of particles of type $i$

We define the current density

$$\vec{J} = \sum_i q_i n_i \langle \vec{v}_i \rangle$$

current through area $\Delta \vec{a}$ is

$$I = \vec{J} \cdot \Delta \vec{a}$$

units of current density is $\frac{\text{charge}}{\text{time}} \frac{1}{\text{area}}$

CGS: esu/s·cm$^2$

MKS: amp/m$^2$

In general the current density can vary in space and time $J(\vec{r}, t)$.

Total current flowing through a surface $S$ is

$$I = \int_S \vec{J} \cdot d\vec{a}$$

Consider a <u>closed</u> surface $S$. If there is a net flux of current flowing out through the surface, then the net total charge inside the surface must be decreasing with time.

$$\boxed{I = \oint_S \vec{J} \cdot d\vec{a} = -\frac{dQ_{encl}}{dt}}$$

conservation of charge

minus since charge inside $S$ is decreasing if $I$ is positive

we can now write

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V dV \; \vec{\nabla} \cdot \vec{J}$$

by Gauss' Theorem
$V$ is volume bounded by $S$

and

$$\frac{dQ_{encl}}{dt} = \frac{d}{dt} \int_V dV \rho = \int_V dV \frac{\partial \rho}{\partial t}$$

so $\displaystyle\int_V dv\, \vec{\nabla}\cdot\vec{J} = -\int_V dv\, \frac{\partial \rho}{\partial t}$ true for *any* volume $V$

If the charge at a point in space is changing it can only be because current is flowing into or out of that point $\Rightarrow$

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}}$$

differential form of conservation of charge.

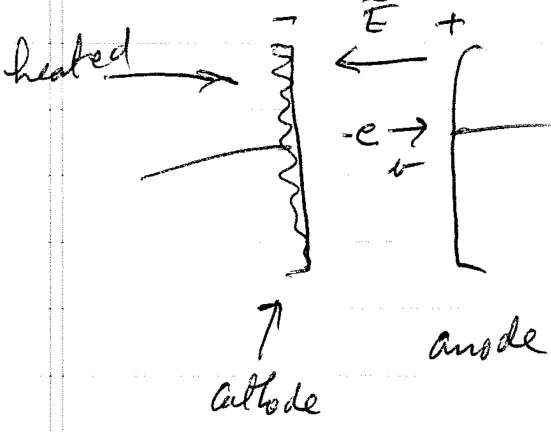
For *electrostatic* situations, the charge density $\rho$ cannot vary in time. Thus, if any currents are flowing in an electrostatic situation it must be that these currents satisfy $\boxed{\vec{\nabla}\cdot\vec{J} = 0}$

This will define the condition of *magnetostatics*

Note: ~~For $\vec{\nabla}\cdot\vec{J} = 0$ it is not necessary~~

Consider a current flowing uniformly in $x$-direction $\vec{J} = J\hat{x}$. For $\vec{\nabla}\cdot\vec{J} = \frac{\partial J}{\partial x} = 0$ it is not necessary that $\rho$ and $\langle v \rangle$ both be constant – only that the product $\rho\langle v\rangle = J$ be constant

vacuum diode – see Fig 4.2 in text




heated cathode emits electrons that travel to anode. Electrons are accelerated by $\vec{E}$ field between cathode and anode
$\Rightarrow$ electrons accelerated
$\Rightarrow v$ is larger at anode than at cathode $\Rightarrow \rho$ is smaller at anode than cathode

in steady state there is steady $J$ from anode to cathode

# Electric Conductivity

electric currents in metals are driven by electric fields.

$\vec{J} = \sigma \vec{E}$

$\sigma$ is the conductivity of the material its value depends on the ~~material~~ properties of the particular ~~metal~~ material. It is large for metals and small for insulators. It may depend on temperature (especially for semiconductors)

We will take $\sigma$ as a scalar, so $\vec{J}$ always points in same direction as $\vec{E}$ For crystalline materials this need not always be so. In that case $\sigma$ can be a matrix

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_y \end{pmatrix}$$

and $\vec{J}$ need not be parallel to $\vec{E}$.

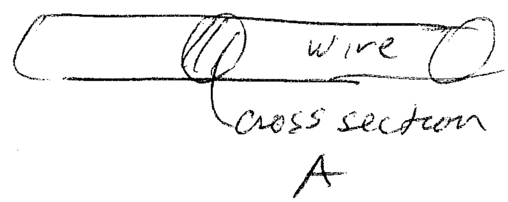
For large electric fields one may also see a non-linear dependence on $\vec{E}$ appear. But for small enough $\vec{E}$ the relation between $\vec{J}$ and $\vec{E}$ is linear.

Sometimes one writes

$$\vec{E} = \rho \vec{J} \quad \text{where} \quad \rho \equiv \frac{1}{\sigma} \text{ is the resistivity}$$

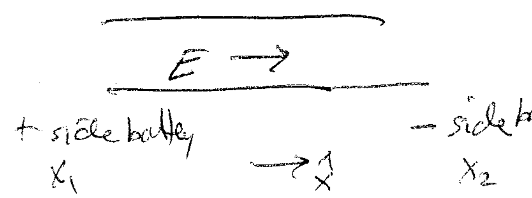
(do not confuse "$\rho$" for resistivity with "$\rho$" for charge density)

Consider now a uniform steady current flowing down a wire. The direction of the current is always tangential to the wire. If $A$ is the perpendicular cross sectional area of the wire then the total current




flowing down the wire is $I = \vec{A} \cdot \vec{J} = AJ$

For a steady flow, the electric field $E$ is uniform




there is then a potential drop down the length $L$ of the wire given by $V = EL$

since

$$\vec{E} = -\vec{\nabla}\phi \quad \text{for } \vec{E} = E\hat{x} \text{ as in picture above}$$

then $$\phi(x_2) - \phi(x_1) = -\int_{x_1}^{x_2} \vec{E} \cdot d\vec{x} = -E(x_2 - x_1)$$

$$= -EL \qquad x_2 - x_1 = L \text{ length of wire}$$

potential drop

$$V \equiv \underset{\substack{\uparrow \\ \text{potential} \\ + \text{ side battery}}}{\phi(x_1)} - \underset{\substack{\uparrow \\ \text{potential} \\ - \text{ side battery}}}{\phi(x_2)} = EL$$

So

$$I = AJ = A\frac{1}{\rho}E = \frac{A}{\rho}\frac{V}{L}$$

$$V = \frac{\rho L}{A} I$$

$$\boxed{V = R I}$$ Ohm's Law, $R = \frac{\rho L}{A}$ is resistance

Resistance scales proportional to length of wire and ~~[illegible]~~ inverse of cross sectional area.

$\rho$ resistivity depends on the particular material but not the geometry. $R$ depends also on the geometry of the conductor.

For a steady current flow $\vec{\nabla}\cdot\vec{J} = 0$
since $\vec{J} = \sigma\vec{E}$, for a uniform material with uniform $\sigma$

$$\Rightarrow \vec{\nabla}\cdot\sigma\vec{E} = \sigma\vec{\nabla}\cdot\vec{E} = 0$$

$\Rightarrow \rho = 0$ no net charge builds up in conductors with steady currents.

Units MKS: $I$ is amps, $V$ is volts, $R$ is ohms
should have $\rho$ is ohms-m, $\sigma$ is $\frac{1}{\text{ohm-m}}$
but often use $\rho$ in ohm-cm, $\sigma$ in $\frac{1}{\text{ohm-cm}}$

one often writes ohm as the symbol $\Omega$.

CGS: $I$ is esu/sec , $V$ is statvolt $= \frac{\text{esu}}{\text{cm}}$ (EL)

$R = \frac{V}{I}$ is $\frac{\text{sec}}{\text{cm}}$ , $\rho = R \cdot \text{cm}$ is sec , $\sigma$ is $\frac{1}{\text{sec}}$

## Physics of conduction in a metal

$$J = \sigma E$$

$$-en\langle v \rangle = \sigma E$$

But if there is an $E$, the electrons experience a force $F = -eE$, so they should accelerate, so how can the average velocity remain constant! Why doesnt $\langle v \rangle$ increase linearly in time as for an accelerated particle?

## Recall from mechanics

a particle of mass $m$ in free fall in the atmosphere. Does the particle keep accelerating as it falls? NO! It reaches a terminal velocity due to air resistance. This can be modeled by the equation

$$ma = m\frac{dv}{dt} = -mg - \alpha v$$

gravitational force down

"friction" due to air resistance

If no friction then $v = -gt$ grows linearly in time as particle accelerates.

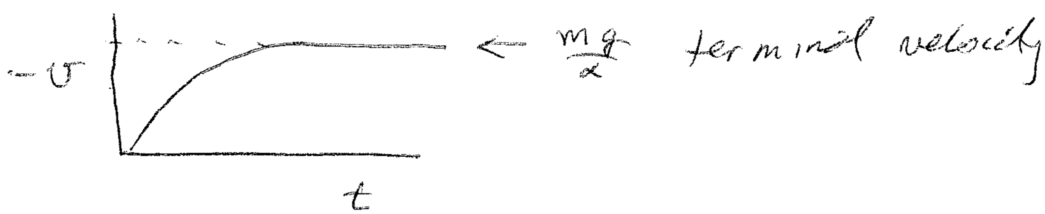
But with friction, what is solution to this differential equation?

$$\frac{dv}{dt} = -\frac{\alpha}{m} v - g$$

$$v(t) = \left(1-e^{-\frac{\alpha}{m}t}\right)\left(-\frac{mg}{\alpha}\right) = -\frac{mg}{\alpha} + \frac{mg}{\alpha} e^{-\frac{\alpha}{m}t}$$

we can check this out

$$\frac{dv}{dt} = -\frac{mg}{\alpha}\left(\frac{\alpha}{m} e^{-\frac{\alpha}{m}t}\right) = -g e^{-\frac{\alpha}{m}t} = -\frac{\alpha}{m} v - g$$




To get the terminal velocity just ~~bass~~ note that at long time $v$ is const so $\frac{dv}{dt} = 0$

so $0 = -mg - \alpha v \Rightarrow v = -\frac{mg}{\alpha}$

What is physics of the air resistance?
Falling body transfers energy to air molecules as it falls. In steady state, increase of energy to particle due to gravitational accel equals loss of energy to collisions with air molecules.
$\Rightarrow v$ constant

friction represents the dissipation of the particles energy due to its transfer to other degrees of freedom (the air molecules)

For electrons in a metal there is a similar frictional term due to transfer of energy of electrons to the ions of the fixed crystal structure.

Simple model

$$\frac{dv}{dt} = -\frac{1}{\tau} v - \frac{e}{m}\vec{E}$$

friction term

driving electric force

$\tau$ is collision time

Then terminal velocity is $v = -\frac{e\tau}{m} E$

current density is

$$-env = \frac{ne^2\tau}{m} E = \sigma E$$

$$\sigma = \frac{ne^2\tau}{m}$$

<u>Drude model for</u> electric conductivity

Originally was that that $\tau$ was due to collisions with stationary ions. In quantum mechanics we find this cannot be true. Correct answer is that $\tau$ is due to transfer of energy of electrons onto vibrations (phonons) of the ions