2) \( \langle \vec{V} \rangle = \frac{1}{m} \sum \vec{m}_i \vec{v}_i \)

It is average velocity of charges that determines the current. In general, the electrons in a wire are travel in random directions with speeds that can be much greater than \( \langle \vec{V} \rangle \). It is only the average velocity that gives rise to a net transport of charge.

Thus for several different types of charged particles \( i \), we have

\[
I = \sum_{i} g_i \cdot m_i \cdot \langle \vec{v}_i \rangle \cdot \Delta A
\]

We define the current density

\[
\vec{J} = \sum_{i} g_i \cdot m_i \cdot \langle \vec{v}_i \rangle
\]

Current through area \( \Delta A \) is

\[
I = \vec{J} \cdot \Delta A
\]

Units of current density is \( \text{charge per unit area} \)

CGS: \( \text{esu/s.cm}^2 \)

MKS: \( \text{amps/m}^2 \)
In general, the current density can vary in space and time $J(F, t)$.

Total current flowing through a surface $S$ is

$$ I = \int_S \mathbf{J} \cdot d\mathbf{A} $$

Consider a closed surface $S$. If there is a net flux of current flowing out through the surface, then the net total charge inside the surface must be decreasing with time,

$$ I = \oint_S \mathbf{J} \cdot d\mathbf{A} = -\frac{dQ_{\text{end}}}{dt} $$

conservation of charge

minus since charge inside $S$ is decreasing if $I$ is positive

we can now write

$$ \oint_S \mathbf{J} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{J} \, dV $$

by Gauss' Theorem

$V$ is volume bounded by $S$

and

$$ \frac{dQ_{\text{end}}}{dt} = \frac{1}{\mathbf{V}} \int_V \nabla \cdot \mathbf{J} \, dV = \int_V \frac{\partial \mathbf{F}}{\partial t} \, dV $$
\[ \mathbf{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \] 

differential form of conservation of charge.

For **electrostatic situations**, the charge density \( \rho \) cannot vary in time. Thus, if any currents are flowing in an electrostatic situation, it must be that these currents satisfy \( \mathbf{\nabla} \cdot \mathbf{J} = 0 \).

This will define the condition of **magneto-statics**.

**Note:** Even \( \mathbf{\nabla} \cdot \mathbf{J} = 0 \) is not necessary.

Consider a current flowing uniformly in \( x \)-direction \( \mathbf{J} = J \mathbf{\hat{x}} \). For \( \mathbf{\nabla} \cdot \mathbf{J} = \frac{\partial J}{\partial x} = 0 \) it is not necessary that \( J \) and \( \mathbf{\nabla} J \) both be constant — only that the product \( \rho (\mathbf{\nabla} \cdot \mathbf{J}) \) be constant.

Vacuum diode — see Fig. 4.2 in text.

**Heated**

\[ \begin{array}{c}
\text{cathode} \\
\text{anode}
\end{array} \]

\[ \begin{array}{c}
\text{e} \\
\text{e} \rightarrow \\
-\text{e} \rightarrow
\end{array} \]

Heated cathode emits electrons that must to anode. Electrons are accelerated by \( E \) field between cathode and anode. Electrons accelerated \( \Rightarrow \) \( v \) is larger at anode than at cathode \( \Rightarrow p \) is smaller at anode than cathode.
Electric Conductivity

electric currents in metals are driven by electric fields.

\[ \vec{J} = \sigma \vec{E} \]

\( \sigma \) is the conductivity of the material, its value depends on the properties of the particular material. It is large for metals and small for insulators. It may depend on temperature (especially for semiconductors).

We will take \( \sigma \) as a scalar, so \( \vec{J} \) always points in some direction as \( \vec{E} \). For crystalline materials this need not always be so. In that case, \( \sigma \) can be a matrix.

\[
\begin{pmatrix}
J_x \\
J_y \\
J_z
\end{pmatrix}
= 
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]

and \( \vec{J} \) need not be parallel to \( \vec{E} \).

For large electric fields one may also see a non-linear dependence on \( \vec{E} \) appear. But for small enough \( \vec{E} \) the relation between \( \vec{J} \) and \( \vec{E} \) is linear.
Sometimes one writes
\[ \mathbf{E} = \rho \mathbf{J} \]
where \( \rho = \frac{1}{\sigma} \) is the resistivity (do not confuse \( \rho \) for resistivity with \( \rho \) for charge density).

Consider now a uniform steady current flowing down a wire. The direction of the current is always tangent to the wire. If \( A \) is the perpendicular cross-sectional area of the wire then the total current flowing down the wire is
\[ I = \mathbf{A} \cdot \mathbf{J} = AJ \]
For a steady flow, the electric field \( \mathbf{E} \) is uniform.

There is then a potential drop
\[ E \rightarrow \]
+ sidebattery \[ x_1 \rightarrow x \]
- sidebattery \[ x \rightarrow x_2 \]
down the length \( L \) of the wire shown by \[ \nabla = EL \]

since \[ \mathbf{E} = -\nabla \phi \]
\[ \mathbf{E} = E \mathbf{x} \]
as in picture above.

Then
\[ \phi(x_2) - \phi(x_1) = - \int_{x_1}^{x_2} E \cdot dx = -E (x_2 - x_1) \]
\[ = -EL \]
\[ x_2 - x_1 = L \text{ length of wire} \]

Potential drop
\[ V = \phi(x_1) - \phi(x_2) = EL \]
\[ I = AJ = A \frac{1}{\rho} E = A \frac{V}{L} \]

\[ V = \frac{PL}{A} \]

\[ V = RI \]  \hspace{1cm} \text{Ohm's Law, } R = \frac{PL}{A} \text{ resistance}

Resistance scales proportionally to length of wire and inversely of cross-sectional area.

Resistivity depends on the particular material but not the geometry. \( R \) depends also on the geometry of the conductor.

For a steady current flow \( \nabla \cdot J = 0 \)

since \( \frac{\partial J}{\partial z} = \sigma \vec{E} \) for a uniform material with uniform \( \sigma \)

\[ \Rightarrow \nabla \cdot \sigma \vec{E} = \sigma \nabla \cdot \vec{E} = 0 \]

\[ \Rightarrow \vec{E} = 0 \] no net charge builds up in conductors with steady currents.

Units: MKS: \( I \) in amperes, \( V \) in volts, \( R \) in ohms should have \( \frac{J}{A} \) in ohms - m, \( \sigma \) is \( \frac{1}{\text{ohm-m}} \) but often use \( \rho \) in ohm-cm, \( \sigma \) is \( \frac{1}{\text{ohm-cm}} \)

one often writes ohm as the symbol \( \Omega \).
CGS: \( F \) in cm/sec, \( V \) in statvolt = \( \frac{esu}{cm} \)  
\[ R = \frac{V}{I} \text{ is sec/cm, } f = \frac{R}{cm} \text{ is sec, } \sigma \text{ is } \frac{1}{\text{sec}} \]

Physics of conduction in a metal

\[ J = \sigma E \]
\[ -e m \langle v \rangle = \sigma E \]

But if there is an \( E \), the electrons experience a force \( F = -eE \), so they should accelerate, so how can the average velocity remain constant? Why doesn't \( \langle v \rangle \) increase linearly in time as for an accelerated particle?

Recall from mechanics

A particle of mass \( m \) in free fall in the atmosphere. Does the particle keep accelerating as it falls? No! It reaches a terminal velocity due to air resistance. This can be modeled by the equation

\[ ma = m \frac{dv}{dt} = -mg - \alpha v \]

gravitational "friction" due to force down and resistance
If no friction then $v = -gt$ grows linearly in time as particle accelerates.

But with friction, what is solution to this differential equation?

$$\frac{dv}{dt} = -\frac{a}{m} v - g$$

$$v(t) = (1 - e^{-\frac{at}{m}}) (-\frac{mg}{a}) = -\frac{mg}{a} + \frac{mg}{a} e^{-\frac{at}{m}}$$

we can check this out:

$$\frac{dv}{dt} = -\frac{mg}{a} (\frac{a}{m} e^{-\frac{at}{m}}) = -g e^{-\frac{at}{m}} = -\frac{mg}{a} v - g$$

\[ -v \quad \begin{array}{c} \text{terminal velocity} \\ \text{t} \end{array} \]

To get the terminal velocity, just note that at long time $v$ is constant so $\frac{dv}{dt} = 0$

So $0 = -mg - av \Rightarrow v = -\frac{mg}{a}$

What is physics of the air resistance?

Fifty body transfers energy to air molecules as it falls. In steady state, increase of energy to particle due to gravitational accel equals loss of energy to collisions with air molecules.

$\Rightarrow \nu$ constant

Fiction represents the dissipation of time.
particles. Energy due to its transfer to other degrees of freedom (the air molecules)

For electrons in a metal there is a similar frictional term due to transfer of energy of electrons to the ions of the fixed crystal structure.

Single model

\[ \frac{dv}{dt} = -\frac{e}{m} \nabla \cdot \mathbf{E} - \frac{e}{m} \mathbf{E} \]

\[ \mathbf{I} = \mathbf{v} \times \mathbf{E} \text{driving electric force} \]

\[ \mathbf{v} \times \mathbf{E} \text{friction term} \]

\[ \mathbf{v} \text{collision time} \]

Then terminal velocity is \( v = -\frac{e}{m} \mathbf{E} \)

Current density \( \mathbf{J} \)

\[ -e \mathbf{v} = \frac{me^2}{m} \mathbf{E} = \sigma \mathbf{E} \]

\[ \sigma = \frac{me^2}{m} \text{Drude model for electric conductivity} \]

Originally was that that \( \mathbf{I} \) was due to collisions with stationary ions. In quantum mechanics we find this cannot be true. Correct answer is that \( \mathbf{I} \) is due to transfer of energy of electrons into vibrations (phonons) of the ions