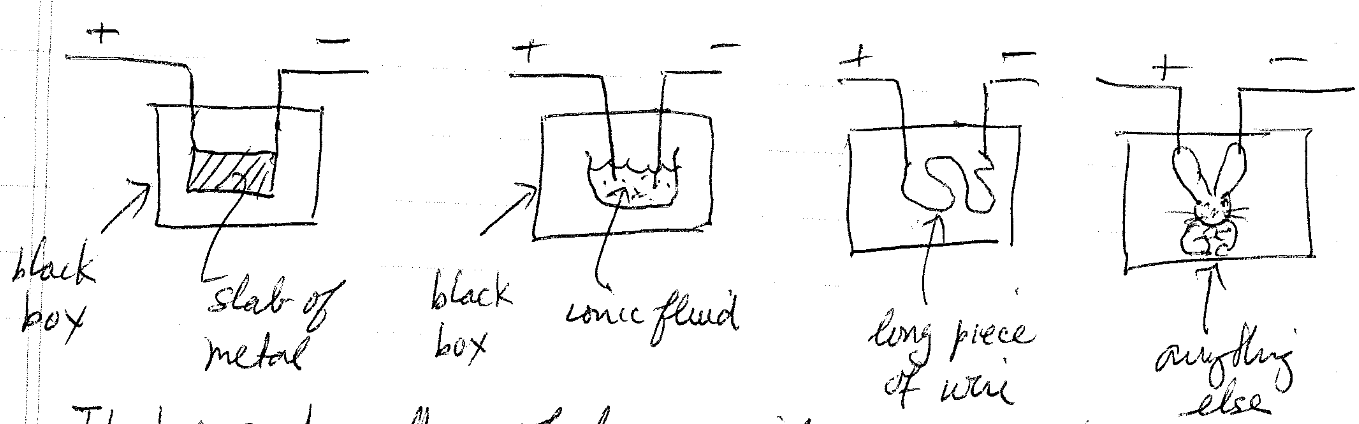
<u>Circuits and circuit elements</u>

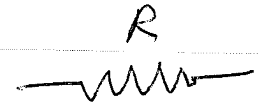
Electric circuits are closed loops of wire (or conductor) in which <u>circuit elements</u> are placed. A circuit element is a "black box" with two terminals - input and output - that is characterized by a well defined relation between current through the element and voltage drop across the element. We do not need to know any details about what is inside the black box as long as we know its current-voltage response.

The simplest element is the <u>resistor</u> R. When a ~~steady (time independent~~ current I flows through the resistor, the voltage drop across the resistor is $V = IR$. This is true even if I is varying in time, ie $V(t) = R\,I(t)$.

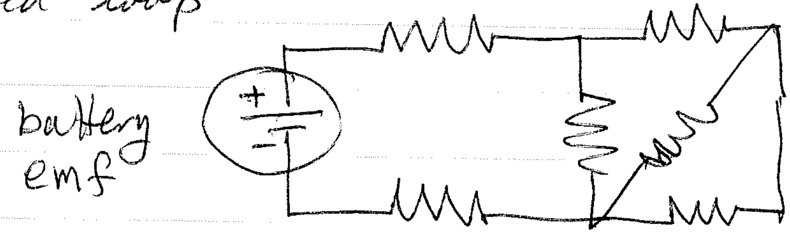
There can be many different physical realizations of a resistor.



It does not matter what is inside the black box as long as we know the resistence across the two terminals of the box.

The symbol for a resistor is 

A circuit connects circuit elements in a closed loop



To determine the behavior of the circuit, ie how much current flows through each element and what is the voltage drop across each element we have <u>Kirchoff's Laws</u>

1) At each node of the circuit (a place where 3 or more wires connect) the sum of currents going into the node equals the sum of currents going out of the node.

This is a consequence of <u>charge conservation</u>. If this were not true, charge would be increasing or decreasing at the node, which should not happen - since node is in conducting wire, the charge would just flow away rather than build up.

2) Sum of potential differences going in order any closed loop of the network must vanish.

This is a consequence of $\oint_C \vec{E}\cdot d\vec{s} = 0$ for all closed loops C.

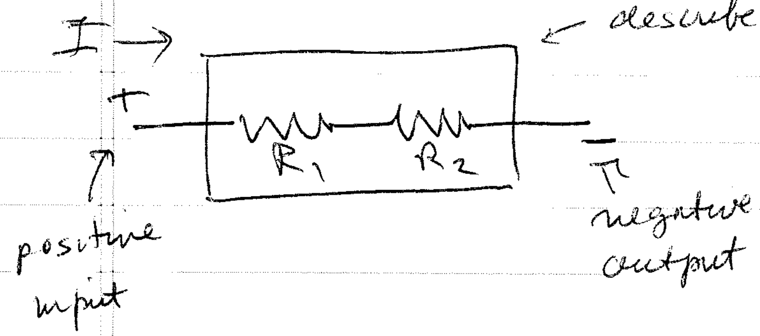
Also: for a resistor, $V = IR$. — linear relation between voltage drop V across resistor and current I through the resistor.

For a network of resistors R_i and batteries, Kirchoff's laws provide the necessary number of linear equations to determine the I_i and V_i for each resistor i given the values R_i and the value of EMF's (electromotive force - or voltage) of each battery in the network.

For complicated networks, it is often possible to replace groups of resistors by an effective resistor that gives the I-V relationship of the entire group. The two most common situations are resistors in series and resistors in parallel

Series

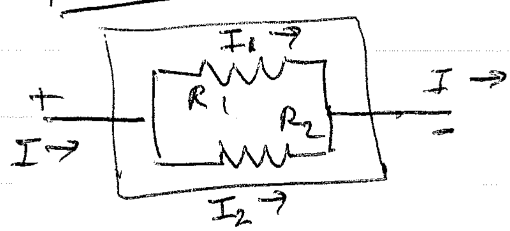
describe two resisters in series as one effective resistor



Suppose current I flows into the first resistor. By Kirchoff law (1), same current I flows through second resistor. The voltage across R_1 is then $V_1 = IR_1$. The voltage across R_2 is $V_2 = IR_2$. The voltage across both resistors is $V = V_1 + V_2 = I(R_1 + R_2) = I R_{eff}$

$$\Rightarrow \boxed{R_{eff} = R_1 + R_2}$$ resisters in series

Parallel

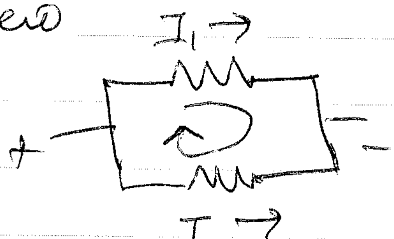


~~By Kirchoff law (2)~~ the ~~voltage across~~

Current I goes in from left, splits into I_1 through R_1 and I_2 through R_2, then combines to give I going out on right.

By Kirchoff (1) $I = I_1 + I_2$ is condition at both left node and right node

By Kirchoff(2) voltage around loop must be zero



$$\Rightarrow \underset{V_1}{I_1 R_1} - \underset{V_2}{I_2 R_2} = 0$$

(−) sign since we defined I_2 going to the right but in computing voltage around loop we go clockwise and so want voltage across R_2 going to the left

So we have that the voltage drop across the two resistors in parallel must be equal

$$V_1 = V_2 \quad \Rightarrow \quad I_1 R_1 = I_2 R_2$$

If $V \equiv V_1 = V_2$ is voltage drop across the pair of resistors then we have

$$V = R_1 I_1 = R_2 I_2 \qquad I = I_1 + I_2$$

define Reff by $V = R_{eff} I$ $\qquad \Rightarrow I_2 = I - I_1$

Solve for I_1 and I_2

$$R_1 I_1 = R_2 I_2 = R_2 (I - I_1)$$

$$(R_1 + R_2) I_1 = R_2 I$$

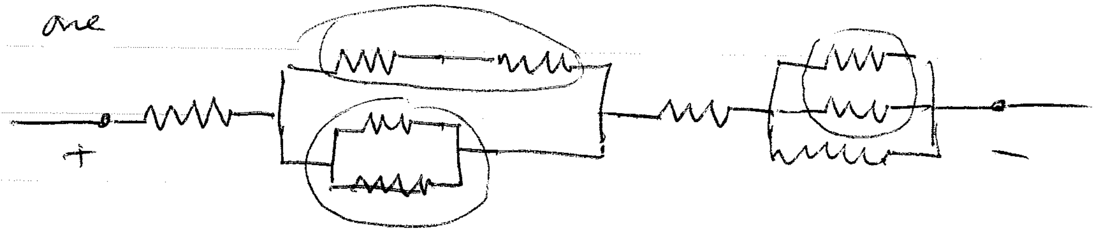
$$I_1 = \frac{R_2}{R_1 + R_2} I$$

similarly $\quad I_2 = I - I_1 = \dfrac{R_1}{R_1 + R_2} I$

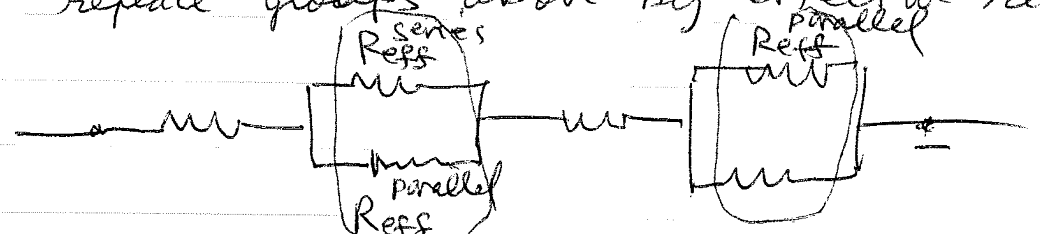
$$R_{eff} = \frac{V}{I} = \frac{I_1 R_1}{I} = \frac{R_2 R_1}{R_1 + R_2} \frac{I}{I}$$

$$\boxed{R_{eff} = \frac{R_2 R_1}{R_1 + R_2}} \quad \text{same as} \quad \boxed{\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

Can then reduce a complex circuit to a simpler one



$\Downarrow$ replace groups above by effective resistors



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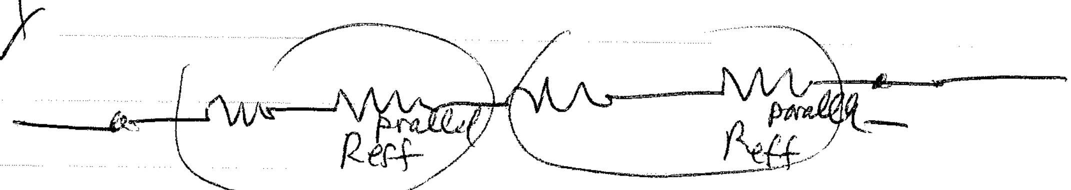

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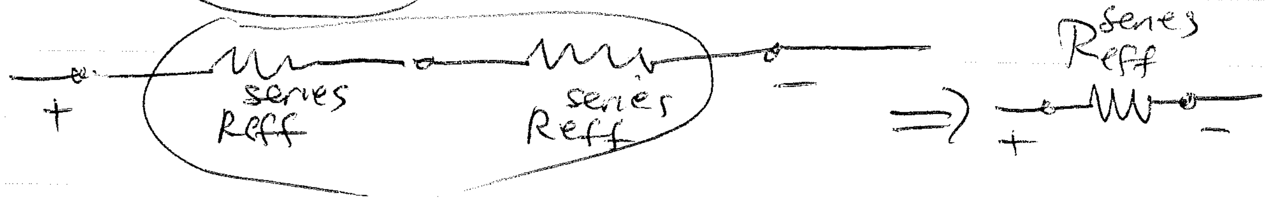
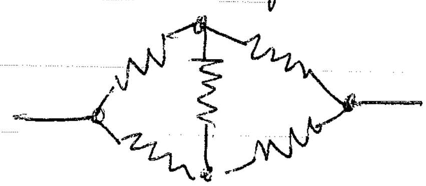
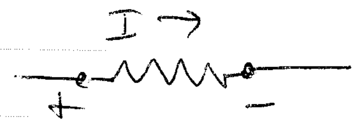


to get the overall current-voltage relation of the circuit.

Sometimes it is not possible to reduce a network into parallel and series sub-groups



In a resistor, current always flows from the terminal at higher potential (+) to the terminal at lower potential (−).



This is because the electric field always points from higher potential to lower potential

$$\phi(+) - \phi(-) = -\int_{-}^{+} \vec{E} \cdot d\vec{s} = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

So $\vec{E}$ is directed from + to − terminal

Energy dissipation in a resistor

If a charge q passes from + terminal of the resistor to the negative terminal, how much energy is released.

$$\text{energy released} = -(\text{work done on } q) = -q(\phi(-) - \phi(+))$$
$$= -q(-V) = qV$$

For current I through resister, $I = \frac{\text{charge}}{\text{time}}$,

the rate of energy released is IV.
This energy released goes into heating the resister. This is the <u>power dissipated</u> by driving a current through the resister.

$$P = IV \quad \text{using } V = IR \text{ we also have}$$
$$P = I^2 R \quad \text{or} \quad P = \frac{V^2}{R}$$

unit of power in MKS is amp-volt $\equiv$ watt $= \frac{\text{joule}}{\text{sec}}$

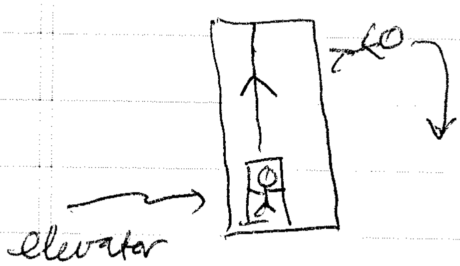
<u>Electromotive force and batteries</u>

If energy is being dissipated as a current flows through a resister, something must be providing that energy. This is the <u>electromotive force</u> $\mathcal{E}$, ~~Most~~ or "emf", or <u>voltage source</u> (~~Most commonly the emf is a battery~~). This gives our second ~~cir~~ type of circuit element. The emf $\mathcal{E}$ is an ~~cir~~ element in which current flows <u>opposite</u> to the average electric field, ie from the ~~(+)~~ negative terminal to the positive terminal, ie from the side with lower electrostatic potential to the side with positive electrostatic potential. The potential difference
$\phi(+) - \phi(-) = \mathcal{E}$ is the emf of the device.

If charge flows opposite to the electric field there must be other forces at work in the emf that results in the net motion of the charges counter to the electric force on them.

In a vande Graaff accellerator these could be mechanical forces. ~~Th~~ The most common type of emf is the familiar "dry cell" or "wet cell" battery in which it is chemical interactions that drive the charges. See section 4.9 of Purcell for some discussion of the physics inside a battery.

<u>A mechanical analogue</u>

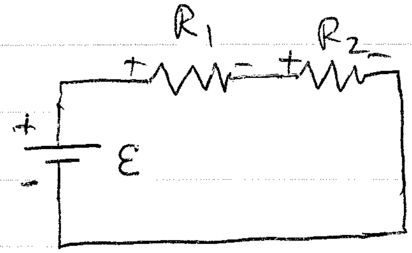


If you jump off a building, the gravitational force will cause you to fall down – air resistance will give you a terminal velocity. As you fall, the ~~cha~~ decrease in your potential energy get dissipated by heating up the air that you collide with. This is like the charge flowing in an electric field down a resistor.

But how do you get to the top of the building in order to jump down? You cannot use the gravitational force to get there since gravity is trying to push you the opposite way, ie downwards. If you want to go upwards you need other forces to move you counter to the force of gravity and bring you from a position of lower potential energy to one of higher potential energy. This could be done with an elevator! The motor of the elevator does the work to drive you against gravity to the top of the building. This is like the charge passing from the lower potential terminal (negative side) to the higher potential terminal (positive side) of an emf or battery. The forces that make this happen are due to physics inside the battery and we henceforth do not concern ourselves with the details!

Circuits with emf's (voltage sources, batteries) and resistors

Resistors in series



suppose $R_1 > R_2$

Is more energy dissipated in R_1 or R_2?

$$-\mathcal{E} + IR_1 + IR_2 = 0 \qquad \text{voltage around loop} = 0$$

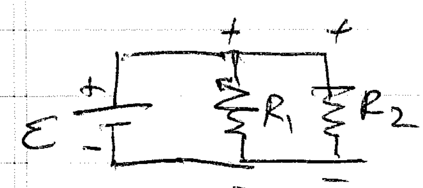
$$\mathcal{E} = I(R_1 + R_2) \Rightarrow I = \frac{\mathcal{E}}{R_1 + R_2}$$

power dissipated in R_1 is $P_1 = I^2 R_1 = \left(\frac{\mathcal{E}}{R_1+R_2}\right)^2 R_1$

power dissipated in R_2 is $P_2 = I^2 R_2 = \left(\frac{\mathcal{E}}{R_1+R_2}\right)^2 R_2$

If $R_1 > R_2$ then $P_1 > P_2$

Resistors in parallel



suppose $R_1 > R_2$

Is more energy dissipated in R_1 or R_2?

$$-\mathcal{E} + I_1 R_1 = 0 \qquad \text{voltage around loop } R_1 = 0$$
$$-\mathcal{E} + I_2 R_2 = 0 \qquad \text{voltage around loop } R_2 = 0$$

$\cancel{\text{so} \quad I_1 = \frac{\mathcal{E}}{R_1} \quad I_2 = \frac{\mathcal{E}}{R_2}}$

$$\Rightarrow \begin{cases} V_1 = I_1 R_1 = \mathcal{E} & \text{voltage drop across } R_1 \\ V_2 = I_2 R_2 = \mathcal{E} & \text{voltage drop across } R_2 \end{cases}$$

power dissipated in R_1 is $P_1 = \frac{V_1^2}{R_1} = \frac{\mathcal{E}^2}{R_1}$

power dissipated in R_2 is $P_2 = \frac{V_2^2}{R_2} = \frac{\mathcal{E}^2}{R_2}$

If $R_1 > R_2$ then $P_2 > P_1$ — opposite as resistors in series!

~~We could also figure out the currents I_1 and I_2 through R_1 and R_2...~~

$$\cancel{I_1 = \frac{\mathcal{E}}{R_1}, \quad I_2 = \frac{\mathcal{E}}{R_2} \quad I = I_1 + I_2}$$

Series config: total power dissipated is

$$P_1 + P_2 = \left(\frac{\mathcal{E}}{R_1+R_2}\right)^2 (R_1+R_2) = \frac{\mathcal{E}^2}{R_1+R_2}$$

Parallel config: total power dissipated is

$$P_1 + P_2 = \frac{\mathcal{E}^2}{R_1} + \frac{\mathcal{E}^2}{R_2} = \mathcal{E}^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

More total power is dissipated in parallel config

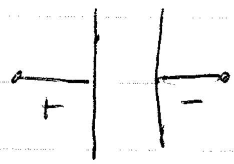
This is because $\frac{(P_1+P_2)\text{parallel}}{(P_1+P_2)\text{series}} = \frac{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\left(\frac{1}{R_1+R_2}\right)}$

$$= \left(\frac{R_1+R_2}{R_1R_2}\right)(R_1+R_2) = \frac{(R_1+R_2)^2}{R_1R_2}$$

$$= \frac{R_1^2 + R_2^2 + 2R_1R_2}{R_1R_2} > 1$$

If $R_1 = R_2 \equiv R$ then $(P_1+P_2)_{\text{parallel}} = \frac{2}{R}$, $(P_1+P_2)_{\text{series}} = \frac{1}{2R}$

Capacitors - another circuit element



typified by parallel plate capacitor

$$V = \frac{Q}{C}$$

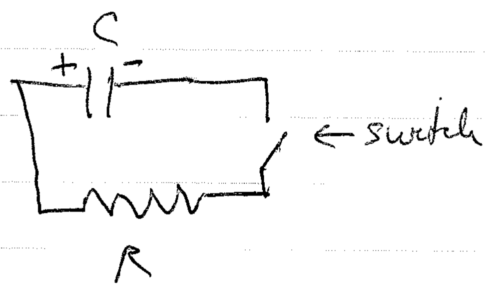
$+Q$ is charge on (+) plate
$-Q$ is charge on (−) plate
$V = \Phi(+) - \Phi(-)$ potential difference

$$\Rightarrow \frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{I}{C}$$

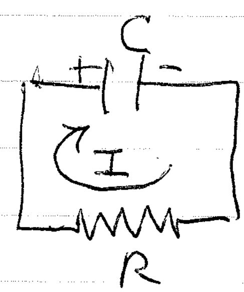
$$\frac{dV}{dt} = \frac{I}{C} \quad \text{or} \quad V(t) - V(t_0) = \frac{1}{C}\int_{t_0}^{t} dt'\, I(t')$$

voltage is not proportional to current as in a resistor, voltage is integral of current over time.

Consider the circuit



at $t=0$ capacitor is charged up with Q. Then switch is suddenly closed. Current will now flow around the loop to discharge the capacitor



since we take voltage drop across C going clockwise around loop, we should take the direction of I also clockwise to be consistent sum of voltage drop around closed loop is zero

$$\Rightarrow \quad V_C + V_R = 0$$

$$\Rightarrow \quad \frac{Q}{C} + IR = 0$$

$$\Rightarrow \quad \frac{Q}{C} + R\frac{dQ}{dt} = 0$$

$$\Rightarrow \quad \frac{dQ}{dt} = -\frac{1}{RC}\, Q$$

solution is $Q(t) = Q(0)\, e^{-t/RC}$

to derive this rewrite above as

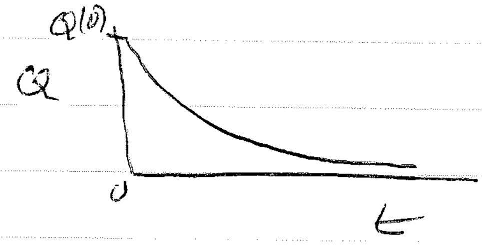
$$\frac{dQ}{Q} = -\frac{dt}{RC}$$

integrate $\displaystyle\int_{Q(0)}^{Q(t)} \frac{dQ}{Q} = -\int_0^t \frac{dt'}{RC} \Rightarrow \ln\left(\frac{Q(t)}{Q(0)}\right) = -\frac{t}{RC}$

$$\Rightarrow \quad \frac{Q(t)}{Q(0)} = e^{-t/RC} \Rightarrow Q(t) = Q(0)\, e^{-t/RC}$$

Since $Q(0) = CV(0)$ we could also write

$$\boxed{Q(t) = CV(0)\, e^{-t/RC}}$$



charge on capacitor decays exponentially in time with a time constant $\boxed{\tau = RC}$

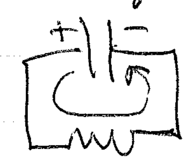
The voltage across the capacitor is

$$V(t) = \frac{Q(t)}{C} = V(0)\, e^{-t/RC}$$

voltage decays exponentially in time

$$I(t) = \frac{dQ(t)}{dt} = -\frac{1}{RC}\, CV(0)\, e^{-t/RC} = -\frac{V(0)}{R}\, e^{-t/RC}$$

I is negative means the current really flows counter clockwise around loop. This is clear since charge should flow from + side of C to − side of C through the wire



magnitude of current decays from its initial value $\frac{V(0)}{R}$ to zero with time constant RC

Units: MKS. $RC = \text{ohm-farad} = \left(\frac{\text{volt}}{\text{amp}}\right)\cdot\left(\frac{\text{coul}}{\text{volt}}\right) = \frac{\text{coul}}{(\text{coul}/\text{sec})}$

$= \text{sec}$ so RC is a time

CGS $RC = \underbrace{\left[\frac{\text{statvolt}}{(\text{esu}/\text{sec})}\right]}_{R}\cdot\underbrace{\{\text{cm}\}}_{C} = \frac{(\text{esu}/\text{cm})}{(\text{esu}/\text{sec})}(\text{cm}) = \text{sec}$

statvolt = esu/cm