__Magnetostatics__ – magnetic phenomena due to __steady__ electric currents, $\vec{\nabla}\cdot\vec{J}=0$

Originally, magnetics – the properties of permanent bar magnets as found in lodestones or used in compass needles – was thought to be independent of electrostatics.

~~[illegible]~~ Until Oersted in ~ 1820 found that magnets interact with electric currents. A compass needle near a current carrying wire is deflected in opposite directions depending on the current in the wire.

Other experiments:

Two straight parallel wires carry currents in the same direction feel a force of attraction $\sim 1/d$ proportional to inverse of distance between them. If currents are in opposite directions they feel ~~repul~~ repulsive force. This happens when there is no net charge on the wires.

A coil (solenoid of wire) with current flowing behaves in most respects just like a compass needle – it orients when placed near a current carrying wire, or another such coil.

The forces between current carrying wires and between compass needles are now understood to all be manifestations of the magnetic force: ~~Steady~~ Flowing electric currents (in fact any moving charge) generates magnetic fields just like a stationary charge (or a moving charge) creates an electric field. Permanent magnets produce magnetic fields because they contain circulating atomic currents like the currents in a wire coil.

In ~~the~~ analogy with electrostatics, we want two things in order to define a theory of magnetostatics

i) what is the magnetic field that results from a particular steady current I

ii) Given a magnetic field, what is the force it exerts on a charged particle?

In electrostatics we had

i) $\vec{E}(\vec{r}) = \int d^3r \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \, \widehat{\vec{r}-\vec{r}\,'}$ Coulomb's law

ii) $\vec{F} = q\vec{E}(\vec{r})$ force on q in magnetic field $\vec{E}$.

In magnetostatics

we consider first the force.

For a charge q moving with velocity $\vec{v}$ in the presence of other ~~static~~ static charges q' and steady currents I, there is a piece of the force on q that is independent of its velocity $\vec{v}$. This is the electric force $q\vec{E}$ where $\vec{E}$ is due to the other charges q'.

Then there is a second contribution to the force on q that is linear in the particles velocity and is perpendicular to the velocity. This force is also proportional to the charge. We can write this force as

$$\vec{F}_L = \frac{q}{c}\vec{v} \times \vec{B} \qquad \text{Lorentz force}$$

↑ cross product since $F_L \perp \vec{v}$ and $\perp \vec{B}$

where $\vec{B}$ is the magnetic field due to the currents I.

Total force is

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

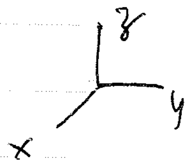
Turns out that this force law holds even more generally when charges are in motion and currents are not steady, ie when we are not in electrostatic-magnetostatic conditions

Recall the cross product $\vec{v} \times \vec{B}$

magnitude $|\vec{v} \times \vec{B}| = |\vec{v}||\vec{B}| \sin\theta$

$\uparrow$ angle between $\vec{v}$ and $\vec{B}$

the direction of $\vec{v} \times \vec{B}$ is given by the <u>Right Hand Rule</u> - fingers of right hand point along $\vec{v}$, then rotate fingers into direction of $\vec{B}$, then thumb points in direction of $\vec{v} \times \vec{B}$.



$\hat{x} \times \hat{z} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$

$\vec{v} \times \vec{B} = -\vec{B} \times \vec{v}$

$\vec{v} \times \vec{B}$ is perpendicular to both $\vec{v}$ and $\vec{B}$

$\vec{v} \times \vec{B} = 0$ if $\vec{v}$ parallel to $\vec{B}$

The cross product $\vec{v} \times \vec{B}$ is what causes the Lorentz force to be perpendicular to the charges velocity. Observing $\vec{F}_L$ and particles velocity $\vec{v}$ determines the direction of $\vec{B}$. (well really only that component of $\vec{B}$ that is not parallel to $\vec{v}$)

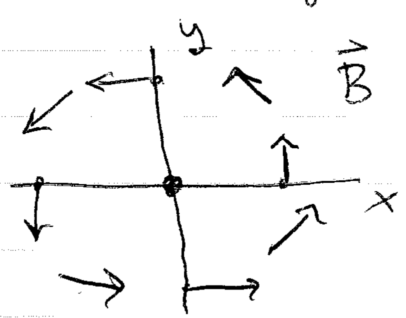
When we started to compute $\vec{E}$ from charges ρ, we started with the simplest case - a point charge q.

When we want to understand how magnetic fields $\vec{B}$ arise from currents $\vec{J}$, we will start with the simplest case of a straight wire carrying a current $\vec{I} = I\hat{z}$ along $\hat{z}$ axis.

Magnitude of magnetic field is $|\vec{B}| = \frac{2I}{rc}$

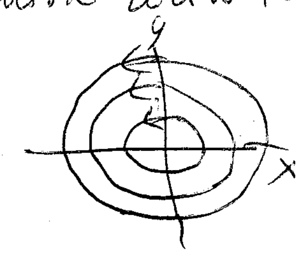
r cylindrical radial coordinate

Direction of $\vec{B}$ is circulating in direction $\hat{z} \times \hat{r} = \hat{\varphi}$ with $\hat{r}$ the cylindrical radial coordinate



$\hat{z}$ out of page

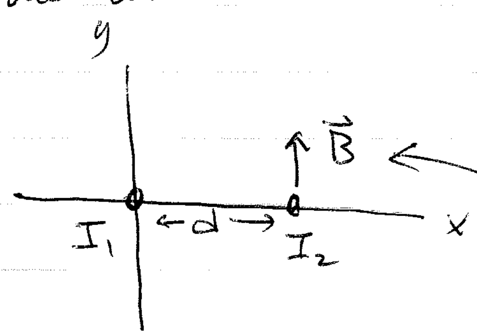
field lines of $\vec{B}$ are circles concentric with the wire



going counter clockwise

Now consider two parallel current carrying wires with currents I_1 and I_2 along $\hat{z}$ direction

image I_1 is out of page along $+\hat{z}$ direction



$\vec{B}$ field due to I_1 at location of I_2

For a charge q flowing in current I_2, the Lorentz force will be

$$\vec{F}_L = q\frac{\vec{v}}{c} \times \vec{B} \qquad \text{assume } q > 0$$

① For $I_2 > 0$, $\vec{v}$ points along $\hat{z}$ parallel currents
② For $I_2 < 0$, $\vec{v}$ points along $-\hat{z}$ anti parallel currents

Case ① parallel currents

$$\vec{F}_L = q\frac{v}{c}\ \hat{z} \times \hat{y}\left(\frac{2I_1}{dc}\right)$$

direction of $\vec{v}$ of q in I_2

direction of $\vec{B}$ from I_1 at location I_2

$$F_L = -\frac{2I_1}{dc}\,\frac{qv}{c}\,\hat{x} \qquad \text{attractive}$$

Now $I_2 = q\,n\,v \cdot A$ ← cross sectional area of wire ②

n: number of charges per volume in wire ②

Force on wire 2 per unit length is

$$\vec{f} = \underbrace{\vec{F}_L}_{\text{force per charge}} \cdot \underbrace{n \cdot A}_{\text{charges per length}} = -\frac{2I_1}{dC}\,\frac{qv}{c}\,nA\,\hat{x}$$

$$\boxed{\vec{f} = -\frac{2I_1I_2}{dc^2}\,\hat{x} \qquad \text{attractive}}$$

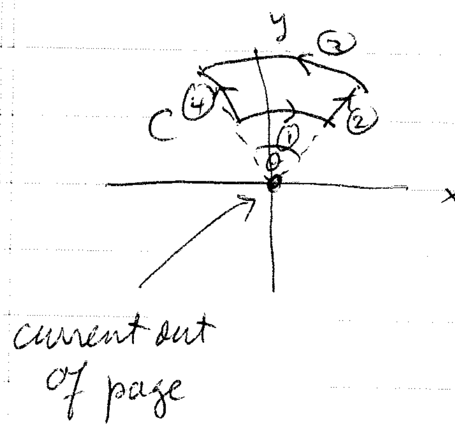
case (2) antiparallel currents

everything is the same except $I_2 \to -I_2$

$$\vec{f} = +\frac{2 I_1 I_2}{d c^2}\,\hat{x} \quad \underline{\text{repulsive}}$$

Ampere's law

Consider a straight infinite wire with $\vec{I} = I\hat{z}$



$$\vec{B} = \frac{2I}{rc}\hat{\varphi}$$

$\hat{\varphi} = \hat{z} \times \hat{r}$ ← cylindrical radial coord

consider line integral

$$\oint_C \vec{B} \cdot d\vec{s} = \frac{2I}{c}\left[\int_1 \vec{B}\cdot d\vec{s} + \int_2 \vec{B}\cdot d\vec{s} + \int_3 \vec{B}\cdot d\vec{s} + \int_4 \vec{B}\cdot d\vec{s}\right]$$

vanish as $\vec{B} \perp d\vec{s} = d\vec{r}$

$$\int_1 \vec{B}\cdot d\vec{s} = \left(\frac{2I}{r_1 c}\right)\cdot(r_1\theta)(-)$$

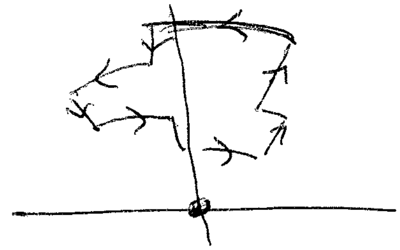
$|\vec{B}|$ at radius r_1 — arclength — since $\vec{B}$ antiparallel to $d\vec{s}$

$$\int_4 \vec{B}\cdot d\vec{s} = \left(\frac{2I}{r_2 c}\right)\cdot(r_2\theta)(+)$$

$|\vec{B}|$ at radius r_2 — arclength — $\vec{B}$ parallel to $d\vec{s}$

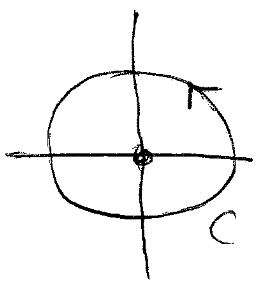
$$\oint_C \vec{B}\cdot d\vec{s} = -\frac{2I}{c}\theta + \frac{2I}{c}\theta = 0$$

Similarly along



And so similarly along any path c that <u>does not</u> enclose the wire

Now consider a path enclosing the wire.

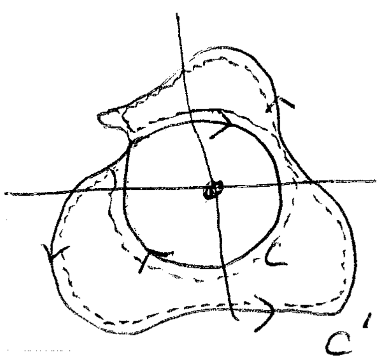


circular path radius r

$$\oint_C \vec{B}\cdot d\vec{s} = \frac{2I}{rc}\,2\pi r$$

$$= \frac{4\pi}{c}\,I$$

Now consider any path enclosing current I



$$\oint_{C+C'} \vec{B}\cdot d\vec{s}$$

is integral along dotted path - does not enclose I so integral = 0.

But $\displaystyle\oint_{C+C'} \vec{B}\cdot d\vec{s} = \oint_{C'} \vec{B}\cdot d\vec{s} + \int_{c} \vec{B}\cdot d\vec{s}$

$= 0$

$$\Rightarrow \oint_{C'} \vec{B}\cdot d\vec{s} = -\int_{c} \vec{B}\cdot d\vec{s} = +\int_{\bar{C}} \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I$$

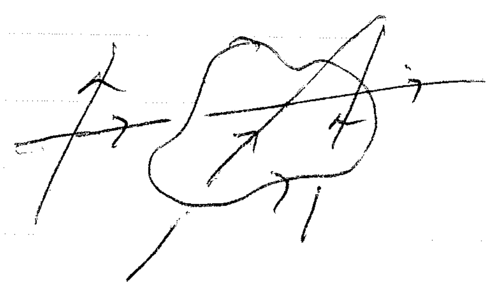
same as c but in opposite direction

So we have shown

$$\oint_C \vec{B}\cdot d\vec{s} = \begin{cases} \frac{4\pi}{c} I & \text{when enclosing wire with current } I \\ 0 & \text{when not enclosing wire } I \end{cases}$$

Now since $\vec{B}$ has no $\hat{z}$ component, the above loop C does not need to lie in flat xy plane, It can wander in z-direction also as long as it closes on itself.

~~By superp~~ By superposition the same will hold true for $\vec{B}$ from a collection of straight current carrying wires, which can be in all directions



$$\oint_C \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I_{encl}$$

total current enclosed by the loop C

Experiments show that the law is even more general than this - Even if current carrying wires are <u>not</u> straight (loops, rings, bent wires) one still has

$$\oint_C \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I_{encl}$$

What is I enclosed?

Let S be any surface bounded by the loop C

then $I_{encl} = \cancel{\oint_S \vec{J}\cdot d\vec{a}}$ $\int_S \vec{J}\cdot d\vec{a}$

current density

flux of $\vec{J}$ through surface bounded by C.

Does it matter what surface S we take? NO!
Consider as an example a circular loop C in xy plane. Let S_1 be the flat circular area bounded by C, and S_2 be the northern hemisphere of a sphere with C as its equator

$$\int_{S_2} \vec{J}\cdot d\vec{a} - \int_{S_1} \vec{J}\cdot d\vec{a} = \int_{S_2} \vec{J}\cdot d\vec{a} + \int_{\bar{S}_1} \vec{J}\cdot d\vec{a}$$

same as S_1 but with normal in opposite direction

$$= \oint_{S_2+\bar{S}_1} \vec{J}\cdot d\vec{a}$$

$$= \int_V \vec{\nabla}\cdot\vec{J}\, dV \quad \text{by Gauss}$$

$= 0$ as $\vec{\nabla}\cdot\vec{J} = 0$ for steady currents!

So,

$$\oint_C \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a}$$

Stokes theorem

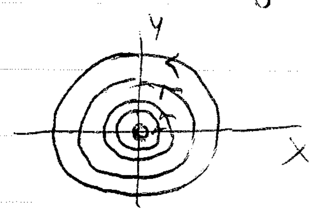
$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a}$$

true for any S $\Rightarrow$ $\boxed{\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}}$ Ampere's Law

$$\boxed{\oint \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{encl}}$$

A second law for $\vec{B}$:

for the straight wire



$\vec{B}$ - field lines are closed ~~circles~~ circles
they do not start nor end anywhere – no sources or sinks

$$\Rightarrow \oint_S \vec{B} \cdot d\vec{a} = 0 \quad \text{any } S$$

flux of B is zero through closed surface

$$\Rightarrow \oint \vec{B} \cdot d\vec{a} = \int dv\, \vec{\nabla} \cdot \vec{B} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

Although our "derivation" came from the field of a straight wire, experiment shows it to be true in general

$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$ $\boxed{\oint_S \vec{B} \cdot d\vec{a} = 0}$ Gauss law for Magnetic fields