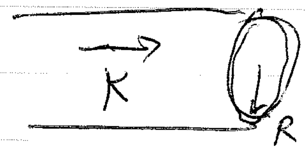
Another example

~~What~~

Conducting cylindrical shell, surface current $\vec{K}$ down length



$\vec{K} = K\hat{z}$ $\quad \vec{B} = B(r)\hat{\varphi}$

$$\oint \vec{B}\cdot d\vec{s} = 2\pi r B(r) = \frac{4\pi}{c} I_{encl}$$

$$2\pi r B(r) = \begin{cases} \frac{4\pi}{c}(2\pi R K) & r > R \\ 0 & R < r \end{cases}$$

$$\vec{B}(r) = \begin{cases} \frac{4\pi R}{c}\frac{K}{r}\hat{\varphi} & r > R \\ 0 & R < r \end{cases}$$

$$\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c}K\hat{\varphi} - 0 = \frac{4\pi}{c}K\hat{\varphi}$$

$\vec{B}(R)$

$$\vec{K} = K\hat{z}, \quad \hat{n} = \hat{r}, \quad \vec{K}\times\hat{n} = K\,\hat{z}\times\hat{r} = K\hat{\varphi}$$

$$\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c}\vec{K}\times\hat{n}$$

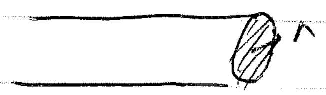
True in general for $\underline{\text{any}}$ surface current even if $\vec{K}$ not uniform:

$$\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c}\vec{K}\times\hat{n}$$

Force on a surface current $\vec{K}$ in a magnetic field

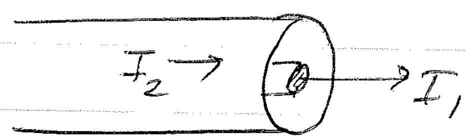
force per area $\quad \vec{f} = \frac{\vec{K}}{c} \times \underbrace{\tfrac{1}{2}(\vec{B}_{above} + \vec{B}_{below})}_{\text{average field at surface}}$

One can use the integral form of Amperes Law to solve generalizations of these three basic problems, for example

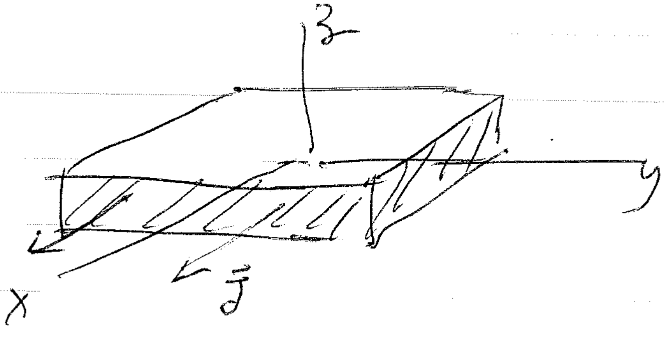


wire of finite thickness with radius dependent current density $J(r)$

or

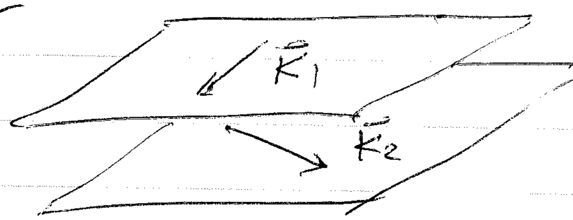


concentric wire and cylindrical shell



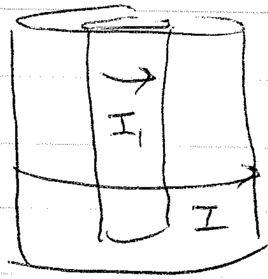
current carrying plane of finite thickness with current density $J(z)$

or



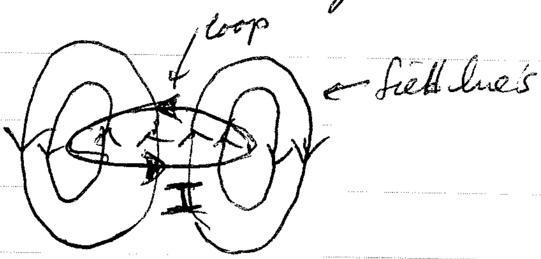
parallel planes with different surface currents

or



~~concentric~~ parallel solenoids

But one could not easily use Ampere's Law directly to solve other "simple" and interesting cases, for example a current carrying circular loop



not enough symmetry to apply $\oint \vec{B} \cdot d\vec{s}$

but we should be able to find some "easy" way to compute $\vec{B}(z)$ on the symmetry axis passing through loops center.

For this we want to develop the magnetostatic equivalent of Coulomb's law, which will allow us to compute $\vec{B}(\vec{r})$ in terms of some well defined integral over specified fixed current distribution. This magnetostatic law is called the Biot-Savart Law.

The vector potential

We have $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$ and $\vec{\nabla} \cdot \vec{B} = 0$

Now in prob 2.16 (Problem Set 4) you showed that for any vector function $\vec{A}$, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Since $\vec{\nabla} \cdot \vec{B} = 0$ we therefore conclude that there must be a vector field $\vec{A}$ such that $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

$\vec{A}$ is called the magnetic vector potential

[This is similar to what happens in electrostatics: since $\vec\nabla\times(\vec\nabla\phi)=0$ for any scalar function ϕ, and since $\vec\nabla\times\vec E=0$, we conclude there must be some ϕ such that $\vec E=-\vec\nabla\phi$]

in terms of $\vec A$, Ampere's Law is then

$$\vec\nabla\times(\vec\nabla\times\vec A) = \frac{4\pi}{c}\vec J$$

We can work out the double curl by using components

$$\vec\nabla\times\vec A = \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\ \frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\ \frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$\begin{aligned}
\vec\nabla\times(\vec\nabla\times\vec A) = \Bigg(&\frac{\partial}{\partial y}\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]-\frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right],\\
&\frac{\partial}{\partial z}\left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]-\frac{\partial}{\partial x}\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right],\\
&\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]-\frac{\partial}{\partial y}\left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]\Bigg)
\end{aligned}$$

$$\begin{aligned}
= \Bigg(&-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z},\\
&-\frac{\partial^2 A_y}{\partial x^2}-\frac{\partial^2 A_y}{\partial z^2}+\frac{\partial^2 A_z}{\partial y\partial z}+\frac{\partial^2 A_x}{\partial x\partial y},\\
&-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z}\Bigg)
\end{aligned}$$

$$\begin{aligned}
= \Bigg(&-\frac{\partial^2 A_x}{\partial x^2}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right),\\
&-\frac{\partial^2 A_y}{\partial x^2}-\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial z^2}+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right),\\
&-\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_z}{\partial y^2}-\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\Bigg)
\end{aligned}$$

$$= \left(-\nabla^2 A_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{A}),\ -\nabla^2 A_y + \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{A}),\ -\nabla^2 A_z + \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{A}) \right)$$

$$= -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

where $-\nabla^2\vec{A} \equiv (-\nabla^2 A_x, -\nabla^2 A_y, -\nabla^2 A_z)$

$\leftarrow$ Laplacian operating on a vector function

So $$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{J}}$$

Ampere's law in terms of the vector potential

Now when we write $\vec{B} = \vec{\nabla}\times\vec{A}$, there is not a unique $\vec{A}$ that satisfies this requirement

[recall, in $\vec{E} = -\vec{\nabla}\phi$ there is similarly not a unique ϕ. If $\phi' = \phi + c$, with c a constant, then also $-\vec{\nabla}\phi' = \vec{E}$]

To see this suppose we have an $\vec{A}$ such that $\vec{\nabla}\times\vec{A} = \vec{B}$. Then construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ where $\chi(\vec{r})$ is <u>any</u> scalar function. Then

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\chi) = \vec{B} + 0$$

$\uparrow$
$= 0$ since $\vec{\nabla}\times(\vec{\nabla}\chi) = 0$ for <u>any</u> scalar χ

So $\vec{A}'$ is also a good vector potential for giving $\vec{B}$.