We can use the freedom in the many choices of $\vec{A}$ to choose one that satisfies $\vec{\nabla}\cdot\vec{A}=0$.

[Suppose we had an $\vec{A}$ such that $\vec{\nabla}\times\vec{A}=\vec{B}$, but $\vec{\nabla}\cdot\vec{A}=f(\vec{r})\neq 0$. Then construct $\vec{A}'=\vec{A}+\vec{\nabla}\chi$. We automatically have $\vec{\nabla}\times\vec{A}'=\vec{B}$. If we want $\vec{A}'$ such that $\vec{\nabla}\cdot\vec{A}'=0$, then we need to find a χ such that $\vec{\nabla}\cdot\vec{A}'=\vec{\nabla}\cdot\vec{A}+\nabla^2\chi=0 \Rightarrow -\nabla^2\chi=f(\vec{r})$. But this is just like solving for electrostatic potential ϕ from charge distribution $\rho=\frac{f}{4\pi}$. There is always a solution! So we <u>can</u> always find an $\vec{A}$ such that $\vec{\nabla}\times\vec{A}=\vec{B}$ <u>and</u> $\vec{\nabla}\cdot\vec{A}=0$]

For such a vector potential that satisfies $\vec{\nabla}\cdot\vec{A}=0$, Amperes law becomes

$$-\nabla^2\vec{A}=\frac{4\pi}{c}\vec{J}$$

or, for example, for the x-component

$$-\nabla^2 A_x=\frac{4\pi}{c}J_x$$

but this is mathematically the same equation as $-\nabla^2\phi=4\pi\rho$. For a $\vec{J}_x$ that is localized to a finite of region of space, we therefore know that the solution for $\vec{A}_x$ can be expressed as the integral

$$A_x(\vec{r})=\frac{1}{c}\int dx'dy'dz'\,\frac{J(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

or, for all components

$$\boxed{\vec{A}(\vec{r}) = \frac{1}{c}\int dx'dy'dz' \; \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

Similar to Coulomb solution for ϕ,

$$\phi(\vec{r}) = \int dx'dy'dz' \; \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

only the expression for $\vec{A}$ is now a vector integral equation,

One can show that the $\vec{A}$ above does satisfy $\vec{\nabla}\cdot\vec{A} = 0$

For a current carrying wire that traces out the path $\mathcal{C}$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int_{\mathcal{C}} d\vec{s}' \, \frac{I}{|\vec{r}-\vec{r}'|}$$

$\vec{r}'$ on curve $\mathcal{C}$
$d\vec{s}'$ points tangential to curve

then

$$\vec{B} = \vec{\nabla}\times\vec{A}(\vec{r}) = \cancel{\frac{1}{c}\int_{\mathcal{C}} d\vec{s}' \, I(\vec{r}') \nabla}$$

$$= \frac{1}{c}\int_{\mathcal{C}} I \; \vec{\nabla}\times\left(\frac{d\vec{s}'}{|\vec{r}-\vec{r}'|}\right)$$

derivatives act only on $\vec{r}$

consider the x-component of $\vec{\nabla} \times \left(\frac{d\vec{s}\,'}{|\vec{r}-\vec{r}\,'|} \right)$

$$\frac{\partial}{\partial y}\left(\frac{ds_z'}{|\vec{r}-\vec{r}\,'|}\right) - \frac{\partial}{\partial z}\left(\frac{ds_y'}{|r-r'|}\right)$$

$$= ds_z' \frac{\partial}{\partial y} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - ds_y' \frac{\partial}{\partial z}\frac{1}{\sqrt{}}$$

$$= ds_z' \frac{(y-y')}{|\vec{r}-\vec{r}\,'|^3} - ds_y' \frac{(z-z')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \frac{1}{|r-r'|^3}\left[d\vec{s}\,' \times (\vec{r}-\vec{r}\,')\right]_x$$

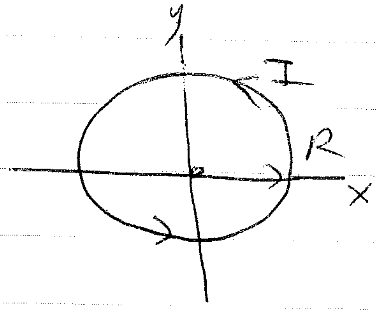
so in general $\vec{\nabla} \times \left(\frac{d\vec{s}\,'}{|\vec{r}-\vec{r}\,'|}\right) = \frac{d\vec{s}\,' \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$

$$= \frac{d\vec{s}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}$$

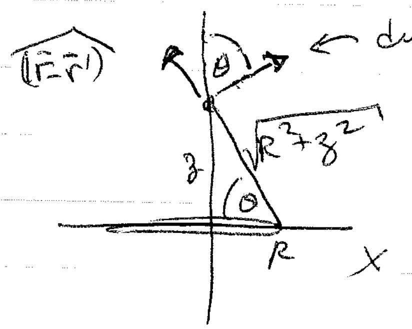
$\widehat{\vec{r}-\vec{r}\,'}$ is unit vector pointing from $\vec{r}\,'$ to $\vec{r}$

$$\boxed{\vec{B}(\vec{r}) = \frac{1}{c}\int_C I \frac{d\vec{s}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}}$$

Biot-Savart Law for magnetic field $\vec{B}$ from current carrying wire that follows the path C.



circular ring in xy plane
What is field at z on $\hat{z}$ axis?



← direction of $d\vec{s}' \times (\vec{r}-\vec{r}')$

$\vec{B}$ will point in $\hat{z}$ direction.
component of $\vec{B}$ in xy plane
cancel out as integrate around loop

$$d\vec{s}' = R\,d\varphi\,\hat{\varphi}$$

$$B_z(z) = \frac{1}{c}\int_0^{2\pi} d\varphi\, R\, \frac{I\cos\theta}{R^2+z^2} \qquad \cos\theta = \frac{R}{\sqrt{R^2+z^2}}$$

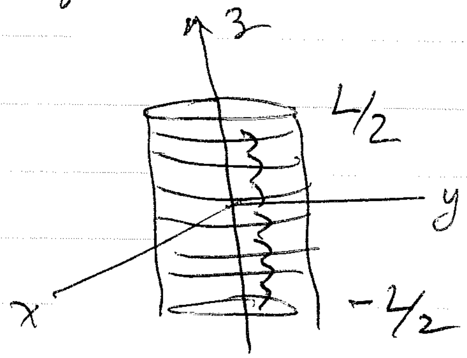
$$= \frac{1}{c}\int_0^{2\pi} d\varphi\, \frac{R^2 I}{(R^2+z^2)^{3/2}} = \frac{2\pi R^2 I}{c\,(R^2+z^2)^{3/2}}$$

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I}{c(R^2+z^2)^{3/2}}\,\hat{z}}$$

at the center of the ring at
$z=0$, $B = \dfrac{2\pi R^2 I}{cR^3}\hat{z}$

$$B = \frac{2\pi I}{cR}\hat{z}$$

~~B~~ Magnetic field along axis of cylinder of finite length



← superposition of many rings
I in wire, N turns / length
$\Rightarrow$ surface current $K = IN$

For a point z above a circular ring was

$$\vec{B}(z) = \frac{2\pi R^2 I}{c(R^2+z^2)^{3/2}}\hat{z}$$

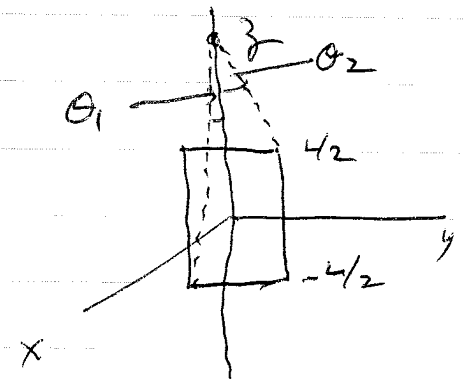
For a point on z axis of solenoid distance to nearest ring is $(z-\frac{L}{2})$, distance to furthest ring is $(z+\frac{L}{2})$

$$\vec{B}(z) = \frac{2\pi R^2 I}{c}\hat{z}\, N\int_{z-\frac{L}{2}}^{z+\frac{L}{2}} dz' \frac{1}{(R^2+z'^2)^{3/2}}$$

$$= \frac{2\pi R^2}{c} IN\hat{z}\left[\frac{z'}{R^2\sqrt{R^2+z'^2}}\right]_{z-\frac{L}{2}}^{z+L/2}$$

$$= \frac{2\pi R^2}{c} IN\hat{z}\left[\frac{(z+\frac{L}{2})}{R^2\sqrt{R^2+(z+\frac{L}{2})^2}} - \frac{(z-\frac{L}{2})}{R^2\sqrt{R^2+(z-\frac{L}{2})^2}}\right]$$

$$\vec{B}(z) = \frac{2\pi I N}{c}\hat{z}\left[\frac{(z+\frac{L}{2})}{\sqrt{R^2+(z+\frac{L}{2})^2}} - \frac{(z-\frac{L}{2})}{\sqrt{R^2+(z-\frac{L}{2})^2}}\right]$$



$$= \frac{2\pi I N}{c}\hat{z}\left[\cos\theta_1 - \cos\theta_2\right]$$

at center of solenoid, $z=0$

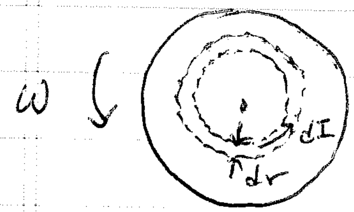
$$\vec{B}(0) = \frac{2\pi I N}{c}\hat{z}\left[\frac{\frac{L}{2}}{\sqrt{R^2+(\frac{L}{2})^2}} - \frac{(-\frac{L}{2})}{\sqrt{R^2+(-\frac{L}{2})^2}}\right]$$

$$= \frac{\pi I N L}{c}\frac{1}{\sqrt{R^2+(\frac{L}{2})^2}}\hat{z}$$

$$= \frac{4\pi I N}{c}\frac{1}{\sqrt{1+(\frac{2R}{L})^2}}\hat{z} \rightarrow \frac{4\pi}{c} I N \hat{z}$$

as $L \rightarrow \infty$

<u>Rowlands Expt</u> rotating disk with surface charge density σ



current flowing in ring of width dr at radius r is

$$dI = \underset{\uparrow \text{ velocity of charges in ring}}{\sigma\, dr\, v} = \sigma\, dr\,(\omega r)$$

$$dI = \underbrace{\sigma \omega r}_{\text{surface current density } K = \sigma \omega r}\, dr$$

$K(r) = \sigma \omega r$ not uniform

Along z axis through center of disk, from the ring of radius r carrying current dI, the contribution to the magnetic field is

$$d\vec{B} = \frac{2\pi r^2\, dI}{c\,(r^2+z^2)^{3/2}}\ \hat{z}$$

Total $\vec{B}$ - field is then

$$\vec{B} = \int d\vec{B} = \hat{z}\int \frac{2\pi r^2\, dI}{c\,(r^2+z^2)^{3/2}}$$

$$= \hat{z}\int_0^R \frac{2\pi r^2}{c\,(r^2+z^2)^{3/2}}\ \sigma\omega r\, dr$$

$$= \hat{z}\ \frac{2\pi\sigma\omega}{c}\int_0^R \frac{r^3}{(r^2+z^2)^{3/2}}\, dr$$

use integrals.wolfram.com

$$= \hat{z}\ \frac{2\pi\sigma\omega}{c}\left[\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right]_0^R$$

$$= \hat{z}\,\frac{2\pi\sigma\omega}{c}\left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - \frac{2z^2}{|z|}\right]$$

$$= \hat{z}\,\frac{2\pi\sigma\omega}{c}\left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2|z|\right]$$

Note: for $R \gg z$ above is $\approx \hat{z}\,\frac{2\pi\sigma\omega}{c} R$

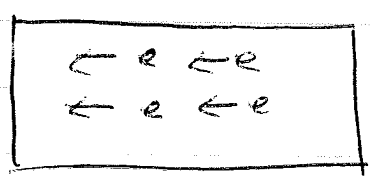
does not look like infinite plane with constant $\vec{K}$ because here $\vec{K}$ is not constant $= \frac{2\pi}{c} K(R)\,\hat{z}$

Hall effect

How do we know that the mobil charges in a conducting metal have negative sign?

1879 E. H. Hall experiment.

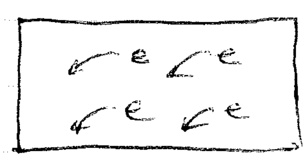
conductor with potential drop across it



$E \rightarrow$

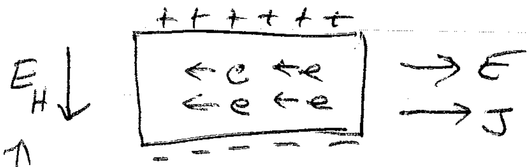
$J \rightarrow$

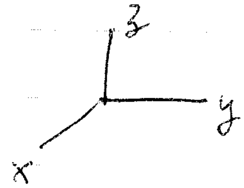
apply $\vec{B}$ out of page



initially, electrons feel Lorentz force from B pushing them downwards

in steady state, electrons will pile up on bottom edge, leaving excess of positive charge on upper edge, creating an electric field pointing downward. Force from this "transverse" or "Hall" electric field exactly balances out force of $\vec{B}$ so electrons move horizontally down wire just as when $B = 0$.





transverse Hall electric field

electron velocity $\vec{v} = -v\hat{y}$ $\qquad \vec{B} = B\hat{x}$

magnetic force $\vec{F}_L = -e\frac{\vec{v}}{c}\times\vec{B}$

$$= +\frac{ev}{c}B\,(\hat{y}\times\hat{x})$$

$$= -\frac{ev}{c}B\,\hat{z}$$

electric force from $\vec{E}_H$ $\qquad \vec{F}_H = +eE_H\,\hat{z}$

$\vec{E}_H = -E_H\hat{z}$

$$eE_H = \frac{ev}{c}B \Rightarrow E_H = \frac{vB}{c}$$

we can then write $\vec{J} = -en\vec{v} \Rightarrow v = \frac{J}{en}$

density of electrons

$$E_H = \frac{JB}{enc} \quad \text{or} \quad \vec{E}_H = +\frac{\vec{J}\times\vec{B}}{enc}$$

If $J = \sigma E$ where σ is ordinary dc conductivity then

$$\frac{E_H}{E} = \frac{\sigma B}{enc}$$

If ~~charge~~ mobile charges had $+e$ charge, then $\vec{E}_H$ would point upwards - ie in opposite direction

Motion of a charged particle in a uniform B-field

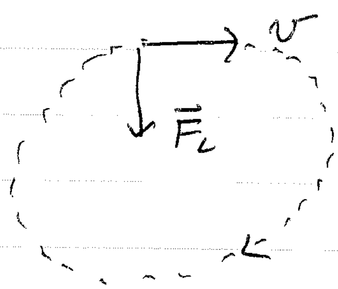
Charge q, velocity $\vec{v} \perp \vec{B}$ initially

$\vec{B} = B\hat{z}$, $\vec{v}$ in xy plane say $\vec{v} = v\hat{x}$

force on q is $\vec{F}_L = q\frac{\vec{v}}{c} \times \vec{B}$

$$= qB\frac{v}{c}(\hat{x} \times \hat{z}) = -q\frac{Bv}{c}\hat{y}$$

force is $\perp \vec{v}$



same as for circular motion!

$$F_L = m a_c = m\frac{v^2}{R}$$

a_c: centripetal acceleration

R is radius of orbit

so $\frac{mv^2}{R} = \frac{qBv}{c}$

$$R = \frac{mv^2 c}{qBv} = \frac{mvc}{qB}$$

~~qB~~ Time to make one orbit is

$$\tau = \frac{2\pi R}{v} = \frac{2\pi}{v}\frac{mvc}{qB} = 2\pi\frac{mc}{qB}$$

angular freq of orbit is $\omega = \frac{2\pi}{\tau} = \frac{qB}{mc}$ "cyclotron" frequency