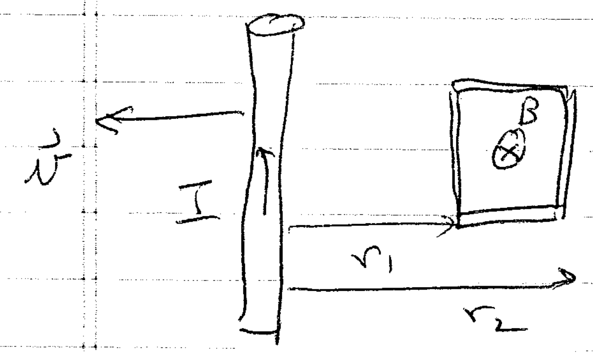
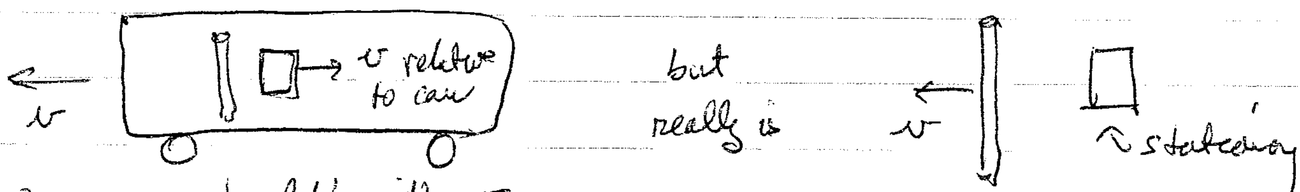
Now consider the case where the loop is staying still, but the infinite wire moves to the left with v



We would expect that the same emf is generated in the loop.

This follows from principle of relativity or, for small v, from <u>Galilean invariance</u>. We cannot determine absolute states of motion at constant velocity



car moving to left with v

Some one in car sees loop moving with v to the right

Some one outside car sees loop stationary and wire moves to left with v.

physics should look the same in either frame of reference

So we expect there to be an emf $\mathcal{E}$ in the loop equal to

$$\mathcal{E} = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \quad \text{Faraday's Law}$$

In this case, Φ is changing because B (at the position of the loop) is changing. This is as opposed to motional emf where B is constant but the location of the loop is changing.

If the loop is not moving in this case, there is no $q \frac{\vec{v}}{c} \times \vec{B}$ driving current around the loop. What then is driving the current?

The forces on charges are magnetic forces and electric forces. If $v=0$ there is no magnetic force. It must be an electric force driving the charges around the loop.

The time varying magnetic field induces an electric field!

electromotive force

$$\mathcal{E} = \oint_C \frac{\vec{F}}{q} \cdot d\vec{s} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{\partial \Phi}{\partial t}$$

Note, the electric field we have here from magnetic induction is very different from what we saw in electrostatics. In electrostatics we always had $\oint_C \vec{E} \cdot d\vec{s} = 0$. But here

$$\oint \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{d\Phi}{dt} \neq 0 \quad \text{in general}$$

(but for static situations, $\frac{d\Phi}{dt} = 0$ and $\oint \vec{E} \cdot d\vec{s} = 0$ as before)

In our first set of examples, B was constant in time and the loop was moving.
In the next example above, the loop is stationary but the source of B is moving. One could also have a case where both loop and source of B are stationary but B is changing in time — for example B is from a solenoid in which the current I is increasing. In all cases it is found

$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt}$$

It does not matter how or why the magnetic flux is changing. This discovery is due to Faraday and the above is known as Faraday's law!

Differential form for Faraday's Law:

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}$$

where S is any surface bounded by the loop C and the direction of $d\vec{a}$ is taken consistent with the circulation around C via the right hand rule.

Now: $$\oint \vec{E} \cdot d\vec{s} = -\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$\parallel$ by Stokes Theorem

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{s} = -\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

This should be true for any surface S bounded by any loop C - there does not need to be a physical wire loop following the path C !

Since it ~~is~~ true for any surface S it must be that

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$$

In MKS units $\nabla \times E = -\dfrac{\partial \vec{B}}{\partial t}$, or $\mathcal{E} = -\dfrac{d\Phi}{dt}$

For static B fields, $\frac{\partial \vec{B}}{\partial t} = 0$, and $\vec{\nabla} \times \vec{E} = 0$ as in statics. But once $\vec{B}$ changes in time, $\vec{\nabla} \times \vec{E} \neq 0$. This means that only in electrostatics is $\vec{E} = -\vec{\nabla}\phi$. This does not hold in general time varying situations

Mutual and Self Inductance

Consider two wire loops C_1 and C_2. Current I_1 flows in loop C_1, producing a magnetic field $\vec{B}_1$.

Let Φ_{21} be the magnetic flux through loop C_2 due to the magnetic field $\vec{B}_1$ from I_1 in loop C_1

$$\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2$$

From Biot-Savart law we know that $\vec{B}_1$ will be proportional to I_1, so Φ_{21} is proportional to I_1. We will call this constant

$$\frac{\Phi_{21}}{I_1} = c M_{21}$$

speed of light

M_{21} is the mutual inductance of loop 2 with respect to loop 1. M_{21} just depends on the geometry of loops 1 and 2.

$$\Phi_{21} = c M_{21} I_1$$

From Faraday's law, if I_1 changes in time slowly (so slowly that $\vec{B}_1$ is the field from I_1 as if we were in a magnetostatic situation)

Then the emf induced in loop 2 due to loop 1

$$\mathcal{E}_{21} = -\frac{1}{c}\frac{d\Phi_{21}}{dt} = -\frac{1}{c}\, c M_{21} \frac{dI_1}{dt}$$

$$\boxed{\mathcal{E}_{21} = -M_{21}\frac{dI_1}{dt}}$$

for $\mathcal{E}$ in (statvolts or esu/cm) and I in esu/s, M_{21} has units s^2/cm.

for $\mathcal{E}$ in volts and I in amps, M_{21} has units of volt-s/amp = ohm-s. In MKS ohm-s $\equiv$ 1 "henry" unit of inductance

Reciprocity Theorem

We can similarly compute the field $\vec{B}_2$ due to a current I_2 in loop C_2, and the flux Φ_{12} it creates through loop C_1. Then

$$\frac{\Phi_{12}}{I_2} \equiv c M_{12}$$

mutual inductance of loop 1 with respect to loop 2

and induced emf in loop 1 due to I_2 in loop 2

$$\mathcal{E}_{12} = -M_{12}\frac{dI_2}{dt}$$

One can show that for any two arbitrary loops, it is always the case that

$$M_{12} = M_{21}$$ this is called the reciprocity theorem

proof:

use the vector potential $\vec{A}$ ~~see~~

recall $\vec{\nabla} \times \vec{A} = \vec{B}$

Stokes theorem

So $\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{S_2} (\vec{\nabla} \times \vec{A}_1] \cdot d\vec{a}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{s}_2$

Now $\vec{A}_1 = \frac{I_1}{c} \oint_{C_1} \frac{d\vec{s}_1}{|\vec{r} - \vec{r}_1|}$ so

$$\Phi_{21} = \oint_{C_2} \vec{A}_1 \cdot d\vec{s}_2 = \frac{I_1}{c} \oint_{C_2} \oint_{C_1} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{|\vec{r}_2 - \vec{r}_1|}$$

Similarly $\Phi_{12} = \frac{I_2}{c} \oint_{C_1} \oint_{C_2} \frac{d\vec{s}_2 \cdot d\vec{s}_1}{|\vec{r}_1 - \vec{r}_2|}$

So $\frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2}$

$$cM_{21} = c\,M_{12} \Rightarrow \boxed{M_{21} = M_{12}}$$

Self inductance

A current I_1 in loop C_1 also results in magnetic flux Φ_{11} throught itself!

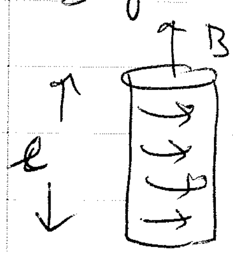
One defines the self-inductance L by

$$\frac{\Phi_{11}}{I_1} = c L_1$$

Then a change in I_1 induces an emf around C_1

$$\mathcal{E}_{11} = -\frac{1}{c}\frac{d\Phi_{11}}{dt} = -L_1\frac{dI_1}{dt}$$

self inductance of a long solenoid



N turns/length

$$B = \frac{4\pi}{c} I N$$

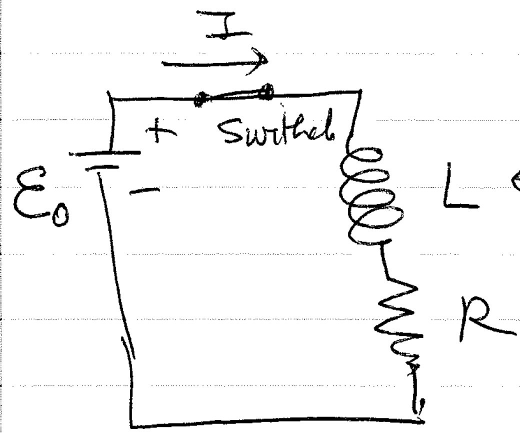
$$\Phi = B \underset{\uparrow\ \text{cross-sectional area}}{A}\ \overbrace{N\ell}^{\#\text{ of turns}}$$

$$\Phi = \frac{4\pi}{c} A N^2 \ell I$$

$$L = \frac{\Phi}{cI} = \frac{4\pi}{c^2} A N^2 \ell$$

$L \propto N^2$ increasing the density of turns greatly increases L

This gives rise to a new type of circuit element called an <u>inductor</u> - one can think of an inductor as a solenoid with a large L



← symbol for an inductor is

consider circuit above

If switch is closed at $t=0$, current starts to flow in loop. This $I(t)$ gives a changing flux through the inductor which creates an emf $-L\frac{dI}{dt}$

Applying Kirchoff's 2nd law around the loop we get

$$\underbrace{\mathcal{E}_0 - L\frac{dI}{dt}}_{} = IR \qquad \text{total emf} = \text{voltage drop across resistor}$$

induced emf acts to oppose the emf $\mathcal{E}_0$ of the battery that drives the current – this is just another example of Lenz's law.

Alternatively we can call $L\frac{dI}{dt}$ = voltage drop across the inductor. So

$$\underset{\text{driving emf}}{\mathcal{E}_0} = \underset{\text{voltage across } L}{L\frac{dI}{dt}} + \underset{\text{voltage across } R}{IR}$$

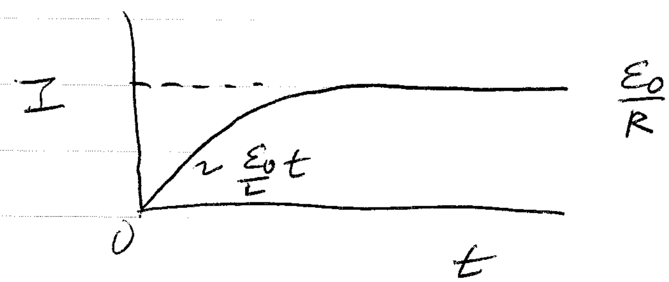
$$\mathcal{E}_0 = L\frac{dI}{dt} + IR$$

after a long time, in steady state, a constant I is flowing. $\Rightarrow \frac{dI}{dt} = 0$ and $I = \frac{\mathcal{E}_0}{R}$

At very short times, when I is ≈ 0, we have

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L} \Rightarrow I = \frac{\mathcal{E}_0}{L}t$$

So current initially increases linearly in time and then saturates to a finite value

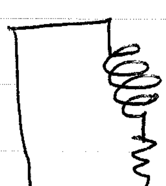


We can solve the differential equation for $I(t)$

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{\mathcal{E}_0}{L}$$

Solution is $I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$

Suppose battery is short circuited when current I_0 is flowing in loop



$$0 = L\frac{dI}{dt} + IR \Rightarrow \frac{dI}{dt} = -\frac{R}{L}I$$

$$I(t) = I_0 e^{-\left(\frac{R}{L}\right)t}$$ decays to zero

Just like the characteristic time constant of a capacitor and resister in series is RC, the characteristic time constant of an inductor and resistor in series is $\cancel{\frac{R}{L}}$. $\frac{L}{R}$.