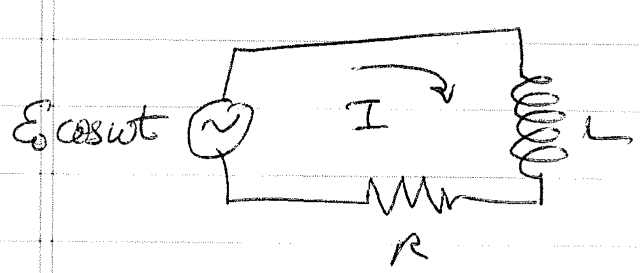
A.C. circuits - add sinusoidal emf



$$\mathcal{E}_0 \cos\omega t = L\frac{dI}{dt} + RI$$

Assume $I(t) = I_0 \cos(\omega t + \varphi)$ oscillates with same freq as source

$$\mathcal{E}_0 \cos\omega t = -LI_0\omega \sin(\omega t + \varphi) + RI_0 \cos(\omega t + \varphi)$$

$$= -LI_0\omega \cos\varphi \sin\omega t - LI_0\omega \sin\varphi \cos\omega t + RI_0 \cos\varphi \cos\omega t - RI_0 \sin\varphi \sin\omega t$$

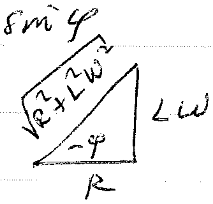
$$\Rightarrow (\mathcal{E}_0 + LI_0\omega \sin\varphi - RI_0 \cos\varphi)\cos\omega t$$

$$= -(LI_0\omega\cos\varphi + RI_0\sin\varphi)\sin\omega t$$

if equal at all times, coefficients of cos and sin must both vanish

$$\Rightarrow LI_0\omega\cos\varphi = -RI_0\sin\varphi$$

$$\boxed{-\frac{L}{R}\omega = \tan\varphi}$$



and

$$\mathcal{E}_0 + (L\omega\sin\varphi - R\cos\varphi)I_0 = 0$$

$$I_0 = \frac{\mathcal{E}_0}{R\cos\varphi - L\omega\sin\varphi}$$

$$\cos \varphi = \frac{R}{\sqrt{R^2+L^2\omega^2}} \qquad \sin \varphi = \frac{-L\omega}{\sqrt{R^2+L^2\omega^2}}$$

$$I_0 = \frac{\mathcal{E}_0 \sqrt{R^2+L^2\omega^2}}{R^2+L^2\omega^2} = \frac{\mathcal{E}_0}{\sqrt{R^2+L^2\omega^2}}$$

Since $\varphi < 0$, current $I(t)$ reaches its max when

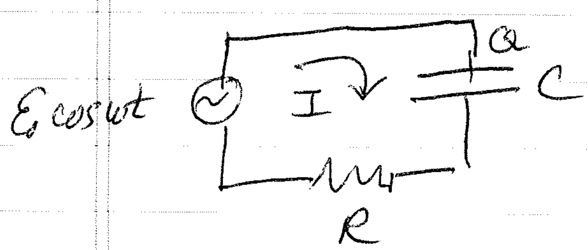
$\omega t + \varphi = 0 \Rightarrow \omega t = |\varphi|$, $t = \frac{|\varphi|}{\omega}$

After $t=0$ when $V(t)$ reaches its peak.

"current lags voltage in inductive circuit"

$$I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2+L^2\omega^2}} \cos\left(\omega t - \arctan\left(\frac{\omega L}{R}\right)\right)$$

Replace inductor by capacitor



$$\mathcal{E}_0 \cos\omega t = \frac{Q}{C} + IR$$

differentiate with respect to time and use $\frac{dQ}{dt} = I$

$$-\mathcal{E}_0 \omega \sin\omega t = \frac{1}{C} I + \frac{dI}{dt} R$$

$$\frac{dI}{dt} + \frac{1}{RC} I + \frac{\mathcal{E}_0 \omega}{R} \sin\omega t = 0$$

Assume solution $I = I_0 \cos(\omega t + \varphi)$

$$= I_0 \cos\varphi \cos\omega t - I_0 \sin\varphi \sin\omega t$$

$$\frac{dI}{dt} = -\omega I_0 \cos\varphi \sin\omega t - \omega I_0 \sin\varphi \cos\omega t$$

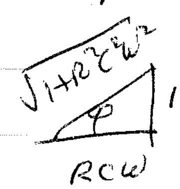
substitute into equ for I and group cos and sin terms

$$\left[-\omega I_0 \cos\varphi - \frac{1}{RC} I_0 \sin\varphi + \frac{\mathcal{E}_0 \omega}{R}\right] \sin\omega t$$

$$+ \left[-\omega I_0 \sin\varphi + \frac{I_0}{RC} \cos\varphi\right] \cos\omega t = 0$$

$= 0$ all t $\Rightarrow$ coeffecients of sin and cos must vanish

$$\Rightarrow \quad \tan\varphi = \frac{1}{RC\omega} \qquad \varphi = \arctan\frac{1}{RC\omega}$$



$$\Rightarrow I_0 = \frac{\mathcal{E}_0 \,\omega}{\omega\cos\varphi + \frac{1}{RC}\sin\varphi}$$

$$= \frac{\mathcal{E}_0\,\omega}{R} \frac{\sqrt{1+R^2C^2\omega^2}}{RC\omega^2 + \frac{1}{RC}} = \frac{\mathcal{E}_0\,\omega C}{\sqrt{1+R^2C^2\omega^2}}$$

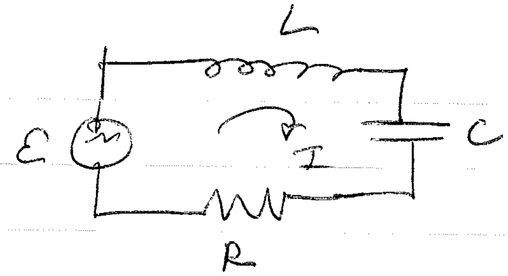
$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}$$

Since $\varphi > 0$, peak of $I(t)$ is at $\omega t = -\varphi$

$t = -\frac{\varphi}{\omega}$ earlier than peak of $V(t)$ at $t = 0$

"current leads voltage in capacitive circuit"

Series LRC circuit — the forced damped harmonic oscillator



$$\mathcal{E} = \mathcal{E}_0 \cos\omega t$$

assume solution $I(t) = I_0 \cos(\omega t + \varphi)$

$$V_L + V_C + V_R = \mathcal{E}$$

$$V_L = L\frac{dI}{dt} = -I_0 \omega L \sin(\omega t + \varphi)$$

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I\,dt = \frac{1}{\omega C} I_0 \sin(\omega t + \varphi)$$

$$V_L + V_C = -\left(\omega L - \frac{1}{\omega C}\right) I_0 \sin(\omega t + \varphi)$$

$$V_R = IR = R I_0 \cos(\omega t + \varphi)$$

$$-\left(\omega L - \frac{1}{\omega C}\right) I_0 \left[\cos\varphi \sin\omega t + \sin\varphi \cos\omega t\right]$$

$$+ R I_0 \left[\cos\varphi \cos\omega t - \sin\varphi \sin\omega t\right]$$

$$= \mathcal{E}_0 \cos\omega t$$

group cos and sin terms

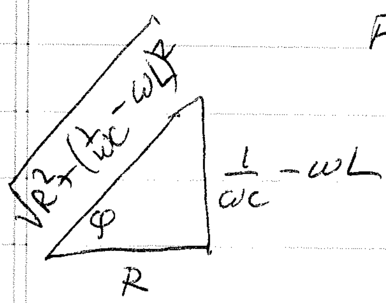
$$\left[-\left(\omega L - \frac{1}{\omega C}\right)\cos\varphi - R\sin\varphi\right] I_0 \sin\omega t$$

$$+ \left[-\left(\omega L - \frac{1}{\omega C}\right)\sin\varphi\, I_0 + R\cos\varphi\, I_0 - \mathcal{E}_0\right]\cos\omega t = 0$$

coeff of sin and cos terms must vanish

$$\Rightarrow \boxed{\tan\varphi = \frac{1}{R}\left(\frac{1}{\omega C} - \omega L\right) = \frac{1}{R\omega C} - \frac{\omega L}{R}}$$

$$\Rightarrow I_0 = \frac{\mathcal{E}_0}{R\cos\varphi - (\omega L - \frac{1}{\omega C})\sin\varphi}$$



$$\cos\varphi = \frac{R}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}}$$

$$\sin\varphi = \frac{\frac{1}{\omega C} - \omega L}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}}$$

$$I_0 = \frac{\mathcal{E}_0 \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$\boxed{I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}}$$

$$I(t) = I_0 \cos(\omega t + \varphi)$$

The greatest current will circulate when the denominator in our expression for I_0 is smallest.

$\Rightarrow$ when $\omega L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$

Recall, $\omega_0 = \frac{1}{\sqrt{LC}}$ was the <u>natural frequency</u> of oscillation when $\mathcal{E}_0 = 0$ and $R = 0$. ie the undamped LC circuit.

This condition of maximum "response" is called resonance

at resonance $I_0 = \dfrac{\mathcal{E}_0}{R}$

$$\tan\varphi = 0 \quad \Rightarrow \varphi = 0$$

current and voltage oscillate in phase

In the limit $\omega \to 0$, $I_0 = \dfrac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$

most important term

$$I_0 \simeq \mathcal{E}_0 \omega C$$

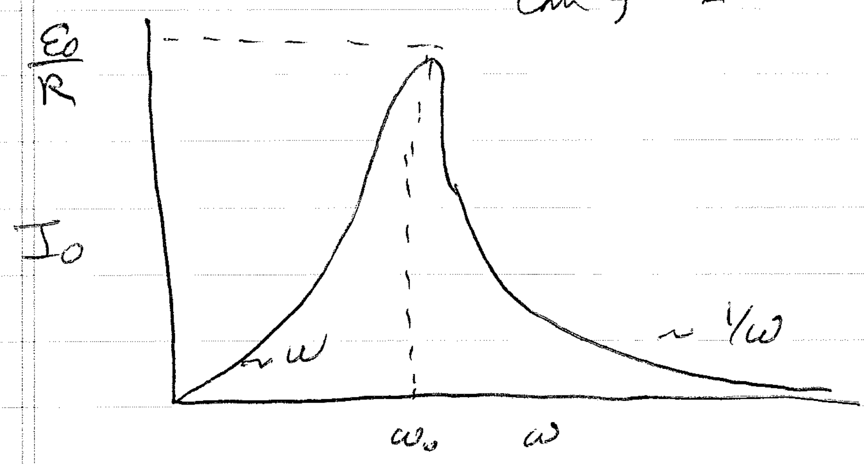
$$\tan\varphi \simeq \infty \Rightarrow \varphi = \frac{\pi}{2}$$

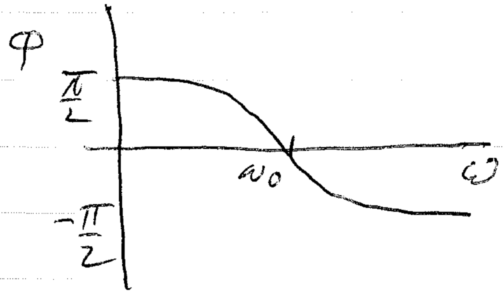
In the limit $\omega \to \infty$, $I_0 = \dfrac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$

most important term

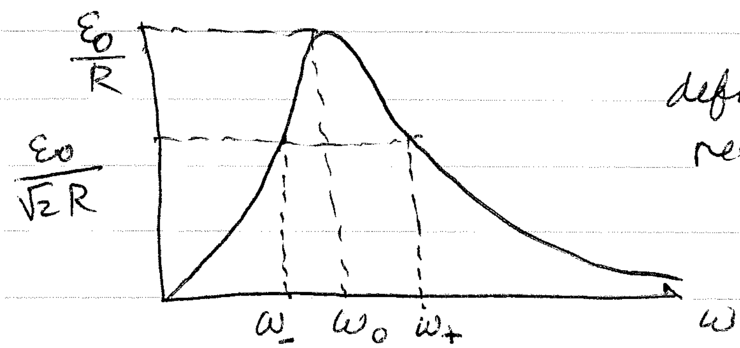
$$I_0 \approx \frac{\mathcal{E}_0}{\omega L}$$

$$\tan\varphi \simeq -\infty \Rightarrow \varphi = -\frac{\pi}{2}$$





As a measure of the width of the resonance, consider the range of frequencies over which I_0 drops to $\frac{1}{\sqrt{2}}$ its resonant value



define the width of the resonant peak as

$$\omega_+ - \omega_- \equiv 2\Delta\omega$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Then we can solve for ω_- and ω_+

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$\omega_\pm$ is when $\left|\omega L - \frac{1}{\omega C}\right| = R$

for ω_- $\Rightarrow$ $\frac{1}{\omega_- C} - \omega_- L = R \Rightarrow \omega_-^2 + \frac{R}{L}\omega_- - \frac{1}{LC} = 0$

for ω_+ $\Rightarrow$ $\omega_+ L - \frac{1}{\omega_+ C} = R \Rightarrow \omega_+^2 - \frac{R}{L}\omega_+ - \frac{1}{LC} = 0$

$$\omega_- = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_+ = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

need (+) solution to keep $\omega_+, \omega_- > 0$

$$\Rightarrow \omega_+ - \omega_- = \left(\frac{R}{2L} + \sqrt{\quad}\right) - \left(\frac{-R}{2L} + \sqrt{\quad}\right) = \frac{R}{L} = 2\Delta\omega$$

$$\boxed{\Delta\omega = \frac{R}{2L}}$$

The sharpness of the resonance is then, in dimensionless units,

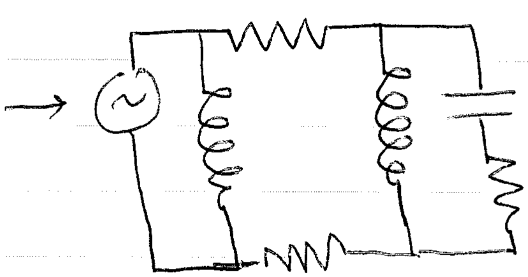
$$\frac{\omega_+ - \omega_-}{\omega_0} = \frac{2\Delta\omega}{\omega_0} = \frac{R}{\omega_0 L} = \frac{1}{Q} \Leftarrow \text{quality factor}$$

If one wants a circuit to have a large responce at a particular frequency ω_0 and a small response elsewhere, one choose L and C so that

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and choose } R \text{ so that} \quad \frac{R}{L} \ll \omega_0$$

Alternating Current Networks

$\mathcal{E}_0 \cos\omega t \rightarrow$



all currents + voltages oscillat with source freq ω

on each link of the network, the current is

$$I_i(t) = I_{i0}\cos(\omega t + \varphi_i)$$

and voltage drop across link is

$$V_i(t) = V_{i0}\cos(\omega t + \theta_i)$$

One uses Kirchoff's laws, together with I-V relationships of each circuit element, to solve for I_i and V_i on each link.