As long as all I_i and V_i oscillate at the same single frequency, circuit analysis is greatly simplified by the use of complex representation based on use of the mathematical identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

one way to prove this is to do a Taylor expansion of each function and then compare real and imaginary terms

It then follows that

$$V_i = V_{i0} \cos(\omega t + \theta_i)$$

$$= \mathrm{Re}\left[V_{i0} e^{i(\omega t + \theta_i)}\right]$$

real part of complex number

$$= \mathrm{Re}\left[V_{i0} e^{i\theta_i} e^{i\omega t}\right]$$

We then call $V_{i0} e^{i\theta_i}$ the "complex" voltage representing the voltage oscillation on link i

The convenience of using this notation is that addition of oscillations with phase shifts just becomes addition of complex numbers:

$$I_1 + I_2 = I_{10} \cos(\omega t + \varphi_1) + I_{20} \cos(\omega t + \varphi_2)$$

do this the trigonometric way:

$$= I_{10} \cos\varphi_1 \cos\omega t - I_{10} \sin\varphi_1 \sin\omega t$$

$$+ I_{20} \cos\varphi_2 \cos\omega t - I_{20} \sin\varphi_2 \sin\omega t$$

$$= (I_{10} \cos\varphi_1 + I_{20} \cos\varphi_2) \cos\omega t - (I_{10} \sin\varphi_1 + I_{20} \sin\varphi_2) \sin\omega t$$

$$I_1 + I_2 = \sqrt{(I_{10}\cos\varphi_1 + I_{20}\cos\varphi_2)^2 + (I_{10}\sin\varphi_1 + I_{20}\sin\varphi_2)^2}\;\cos(\omega t + \delta)$$

$$\text{where } \tan\delta = \frac{(I_{10}\sin\varphi_1 + I_{20}\sin\varphi_2)}{I_{10}\cos\varphi_1 + I_{20}\cos\varphi_2}$$

Now do it the complex number way:

$$I_1 + I_2 = \mathrm{Re}\left[I_{10} e^{i\varphi_1} e^{i\omega t} + I_{20} e^{i\varphi_2} e^{i\omega t} \right]$$

$$= \mathrm{Re}\left[(I_{10} e^{i\varphi_1} + I_{20} e^{i\varphi_2}) e^{i\omega t} \right]$$

$$= \mathrm{Re}\left[\left((I_{10}\cos\varphi_1 + I_{20}\cos\varphi_2) + i\,(I_{10}\sin\varphi_1 + I_{10}\sin\varphi_2) \right) e^{i\omega t} \right]$$

For a complex number $z = x + iy$

$$z = \sqrt{x^2+y^2}\, e^{i\delta} \quad \text{with } \tan\delta = \frac{y}{x}$$

$$I_1 + I_2 = \mathrm{Re}\left[\sqrt{(I_{10}\cos\varphi_1 + I_{20}\cos\varphi_2)^2 + (I_{10}\sin\varphi_1 + I_{20}\sin\varphi_2)^2}\; e^{i\delta} e^{i\omega t} \right]$$

gives same result as trig method.

So when we apply Kirchoff's laws to an ac circuit, we just apply it to addition of complex numbers

$$\sum_{in} I_{i\,in} = \sum_{out} I_{i\,out}$$

$$\sum_{loop} V_i = 0$$

Moreover, the relation between I_i and V_i for any circuit element is easily expressed in terms of this complex notation.

<u>Resistor</u> : $V = IR$

$$\Rightarrow \quad \text{For} \quad I = I_0 \cos(\omega t + \varphi) = \text{Re}\left[I_0 e^{i\varphi} e^{i\omega t} \right]$$
$$V = I_0 R \cos(\omega t + \varphi) = \text{Re}\left[I_0 R e^{i\varphi} e^{i\omega t} \right]$$

complex current $I = I_0 e^{i\varphi}$
Complex voltage $V = R I_0 e^{i\varphi}$ or $V = RI$

<u>Inductor</u> $\quad V = L \dfrac{dI}{dt}$

$$\text{For} \quad I = I_0 \cos(\omega t + \varphi) = \text{Re}\left[I_0 e^{i\varphi} e^{i\omega t} \right]$$
$$V = L \frac{dI}{dt} = -\omega L I_0 \sin(\omega t + \varphi) = \text{Re}\left[i\omega L I_0 e^{i\varphi} e^{i\omega t} \right]$$

complex current $I = I_0 e^{i\varphi}$
complex voltage $V = i\omega L I_0 e^{i\varphi}$ or $V = i\omega L I$

<u>Capacitor</u>

$$V = \frac{Q}{C} \Rightarrow C \frac{dV}{dt} = I \quad \text{or} \quad V = \frac{1}{C} \int dt\, I$$

$$\text{For} \quad I = I_0 \cos(\omega t + \varphi) = \text{Re}\left[I_0 e^{i\varphi} e^{i\omega t} \right]$$

$$V = \frac{1}{\omega C} I_0 \sin(\omega t + \varphi) = \text{Re}\left[\frac{-i}{\omega C} I_0 e^{i\varphi} e^{i\omega t} \right]$$

complex current $I = I_0 e^{i\varphi}$
Complex voltage $V = \dfrac{-i}{\omega C} I_0 e^{i\varphi}$ or $V = \dfrac{-i}{\omega C} I$

So the I-V relations of inductors and capacitors instead of being differential relations become just multiplication of complex numbers!

resistor	$V = RI$
inductor	$V = i\omega L\, I$
capacitor	$V = \frac{-i}{\omega C}\, I$

In this complex representation, the ratio V/I of a circuit element is called its <u>impedance</u> Z. It's inverse $\frac{I}{V}$ is called its <u>admittance</u> Y

$$Y = \frac{1}{Z}$$

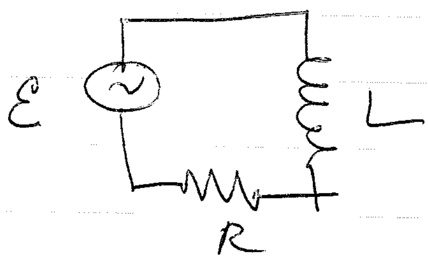
	impedance Z	admittance Y
resistor	R	$\frac{1}{R}$
inductor	$i\omega L$	$\frac{-i}{\omega L}$
capacitor	$\frac{-i}{\omega C}$	$i\omega C$

$$V = ZI \quad \text{or} \quad I = YZ$$

Examples

(1)

$\mathcal{E} = \mathcal{E}_0 \cos\omega t$



$$\mathcal{E} = V_L + V_R$$

$$\mathcal{E}_0 = Z_L I + Z_R I \qquad \text{complex}$$

$$= i\omega L I + R I$$

$$I = \frac{\mathcal{E}_0}{R + i\omega L}$$

$$= \frac{\mathcal{E}_0 (R - i\omega L)}{R^2 + \omega^2 L^2}$$

for real a, b, use $a + ib = \sqrt{a^2+b^2}\, e^{i\delta}$

$\tan\delta = \frac{b}{a}$

$$= \frac{\mathcal{E}_0}{R^2+\omega^2 L^2} (R^2 + \omega^2 L^2)^{1/2} e^{-i\delta}$$

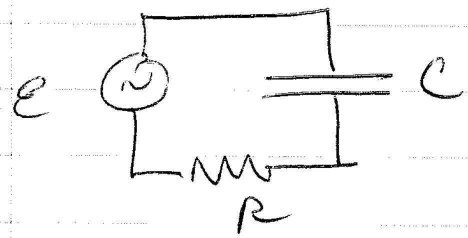
$$\tan\delta = \frac{+\omega L}{R}$$

$$I(t) = \mathrm{Re}\left[\frac{\mathcal{E}_0}{\sqrt{R^2+\omega^2 L^2}} e^{i\delta} e^{i\omega t} \right]$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2+\omega^2 L^2}} \cos(\omega t - \delta)$$

Same answer we found earlier

(2)



$$\mathcal{E} = \mathcal{E}_0 \cos \omega t$$

$$\mathcal{E}_0 = Z_C I + Z_R I$$
$$= \frac{-i}{\omega C} I + R I$$

$$I = \frac{\mathcal{E}_0}{R - \frac{i}{\omega C}} = \frac{\mathcal{E}_0 \left(R + \frac{i}{\omega C}\right)}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$R + \frac{i}{\omega C} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}\, e^{i\delta}$

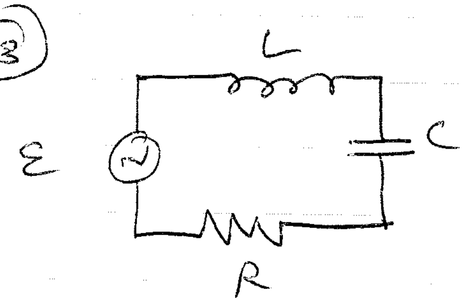
$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{i\delta} \quad \text{where } \tan\delta = \frac{1}{\omega R C}$$

$$I(t) = \mathrm{Re}\left[\frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{i\delta} e^{i\omega t}\right]$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos(\omega t + \delta)$$

same result as earlier

3



$$\mathcal{E} = \mathcal{E}_0 \cos\omega t$$

$$\mathcal{E}_0 = Z_L I + Z_C I + Z_R I$$

$$= i\omega L I - \frac{i}{\omega C} I + R I$$

$$I = \frac{\mathcal{E}_0}{R + i(\omega L - \frac{1}{\omega C})} = \frac{\mathcal{E}_0 \left(R - i(\omega L - \frac{1}{\omega C})\right)}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{i\delta}$$

where $\tan\delta = \dfrac{-\omega L + \frac{1}{\omega C}}{R}$

$$I(t) = \mathrm{Re}\left[\frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{i\delta} e^{i\omega t}\right]$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \cos(\omega t + \delta)$$

$$\tan\delta = \frac{\frac{1}{\omega C} - \omega L}{R}$$

same answer as found earlier!

Power and Energy

If voltage across resistor R is $V_0 \cos \omega t$ then the instantaneous power, i.e. the rate at which energy is being dissipated is

$$P(t) = I^2 R = \frac{V_0^2 \cos^2 \omega t}{R}$$

The *average* power, averaged over one cycle is

$$\bar{P} = \frac{1}{T} \int_0^T dt \, \frac{V_0^2}{R} \cos^2 \omega t$$

where $T = \frac{2\pi}{\omega}$ is one period of oscillation

$$= \frac{1}{2} \frac{V_0^2}{R} \quad \left(\text{since average of } \cos^2 \omega t \text{ is } \tfrac{1}{2}\right)$$

One defines the root-mean-square (rms) value of the voltage by

$$V_{rms} = \left[\frac{1}{T} \int_0^T dt \, (V(t))^2 \right]^{1/2}$$

$$= \left[\frac{1}{T} \int_0^T dt \, V_0^2 \cos^2 \omega t \right]^{1/2}$$

$$= \left[\frac{1}{2} V_0^2 \right]^{1/2}$$

$$\boxed{V_{rms} = \frac{1}{\sqrt{2}} V_0} = \frac{\text{Amplitude of oscillation}}{\sqrt{2}}$$

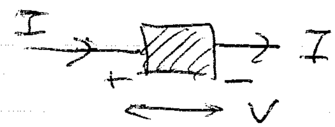
In terms of V_{rms},

$$\bar{P} = \frac{1}{2}\frac{V_0^2}{R} = \frac{V_{rms}^2}{R}$$

When one gives the values of an ac voltage or current, it is customary to give the rms value $V_{rms} = \frac{1}{\sqrt{2}} V_0$ or $I_{rms} = \frac{1}{\sqrt{2}} I_0$, rather than the amplitude of the oscillation V_0 or I_0

In general, the instantaneous power delivered to any circuit element is

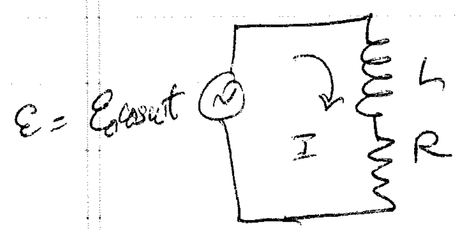
$P(t) = V(t) I(t)$ where $V(t)$ is voltage drop across the element



(this follows since ~~QV is the work done moving Q across the voltage drop V~~ so the ~~work~~ power ~~done~~ to ~~move~~ one charge Q is $Q\vec{v}\cdot\vec{E}$, $\vec{v}$ is velocity, $\vec{E}$ is electric field. Total power done so $(nAL)Q\vec{v}\cdot\vec{E}$ where A is area, L length, n charge density of element so nAL is number of charges.

$$(nALQ\vec{v}\cdot\vec{E}) = nALQv\left(+\frac{V}{L}\right)$$
$$= nAQvV = IV$$)

Consider the LR circuit



we found $I(t) = I_0 \cos(\omega t + \varphi)$

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2+\omega^2L^2}} \qquad \tan\varphi = -\frac{\omega L}{R}$$

The power delivered to the LR combination is

$$P(t) = \mathcal{E}(t)\, I(t) = \mathcal{E}_0 I_0 \cos\omega t \cos(\omega t + \varphi)$$

$$= \mathcal{E}_0 I_0 (\cos\omega t)(\cos\omega t \cos\varphi - \sin\omega t \sin\varphi)$$

$$= \mathcal{E}_0 I (\cos\varphi \cos^2\omega t - \sin\varphi \sin\omega t \cos\omega t)$$

$$= \mathcal{E} I \left(\cos\varphi \cos^2\omega t - \sin\varphi \frac{\sin 2\omega t}{2}\right)$$

when we integrate over one period of oscillation, $\cos^2\omega t$ averages to $\frac{1}{2}$ while $\sin 2\omega t$ averages to zero.

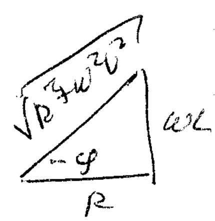
$$\bar{P} = \frac{1}{2}\mathcal{E}_0 I_0 \cos\varphi = \mathcal{E}_{rms} I_{rms} \cos\varphi$$

In this circuit, all the energy dissipated is dissipated in the resistor R.

$$V_R = IR = I_0 R \cos\omega t$$

$$P_R(t) = I V_R = I_0^2 R \cos^2\omega t$$

$$\bar{P}_R = \frac{1}{2} I_0^2 R$$



Compare to $\bar{P} = \frac{1}{2}\mathcal{E}_0 I_0 \cos\varphi = \frac{1}{2}\sqrt{R^2+\omega^2L^2}\; I_0^2 \frac{R}{\sqrt{R^2+\omega^2L^2}} = \frac{1}{2} I_0^2 R$

wherae the energy dissipated in the inductr is

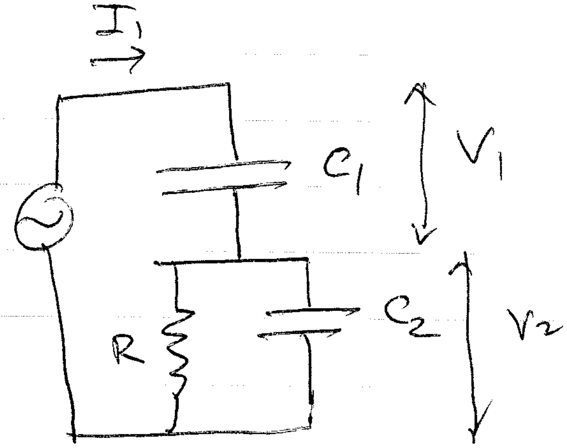
$$V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t+\varphi)$$

$$P_L(t) = I V_L = -\omega L I_0^2 \sin(\omega t+\varphi)\cos(\omega t+\varphi)$$

$$= -\tfrac{1}{2}\omega L I_0^2 \sin \underbrace{2(\omega t+\varphi)}_{\text{averages to zero}}$$

$$\bar{P}_L = 0$$

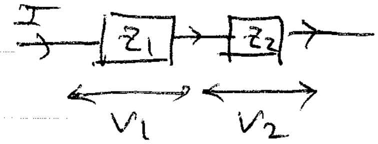
so all energy is dissipated in R.



for parallel elements $\dfrac{1}{Z_{eff}} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$

for series elements $Z_{eff} = Z_1 + Z_2$

series



we have $V = V_1 + V_2$

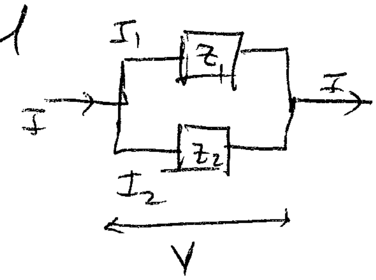
$$= IZ_1 + IZ_2$$

$$= I(Z_1+Z_2)$$

$$= I Z_{eff}$$

$$\Rightarrow Z_{eff} = Z_1 + Z_2$$

parallel



we have $V = I_1 Z_1$

$$V = I_2 Z_2$$

$$I_1 + I_2 = I$$

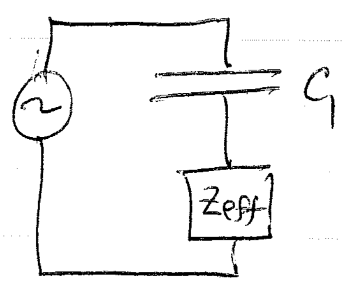
$$\Rightarrow \frac{V}{Z_1} + \frac{V}{Z_2} = \frac{V}{Z_{eff}}$$

$$\Rightarrow \frac{1}{Z_{eff}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

we have $\quad Z_R = R \ , \quad Z_{C_2} = \frac{-i}{\omega C_2} \ , \quad Z_{C_1} = \frac{-i}{\omega C_1}$

replace R and C_2 in parallel by

$$Z_{eff} = \frac{Z_R Z_{C2}}{Z_R + Z_{C2}} = \frac{R\left(-\frac{i}{\omega C_2}\right)}{R + \left(\frac{-i}{\omega C_2}\right)}$$



now replace C_1 and Z_{eff} in series by

$$Z_{tot} = Z_{eff} + \frac{-i}{\omega C_1}$$

$$= \frac{\frac{-iR}{\omega C_2}}{R - \frac{i}{\omega C_2}} - \frac{i}{\omega C_1}$$

$$= \frac{-iR}{\omega C_2 R - i} - \frac{i}{\omega C_1}$$

$$= \frac{R}{1 + i\omega C_2 R} - \frac{i}{\omega C_1}$$

$$= \frac{R(1 - i\omega C_2 R)}{1 + \omega^2 C_2^2 R^2} - \frac{i}{\omega C_1}$$

$$Z_{tot} = \frac{R}{1+\omega^2 C_2^2 R^2} - i\left(\frac{\omega C_2 R^2}{1+\omega^2 C_2^2 R^2} + \frac{1}{\omega C_1} \right) = Z_1 + i Z_2$$

Z_1, Z_2 real

If the voltage is $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$, then the current is

$$\cancel{I_1 \approx \mathcal{E}_0 \cos \omega t} \quad I_1 = \frac{\mathcal{E}_0}{Z_{tot}} = I_0 \cos(\omega t + \varphi)$$

where $I_0 = \dfrac{\mathcal{E}_0}{(Z_1^2 + Z_2^2)^{1/2}}$ $\quad \tan\varphi = \dfrac{Z_2}{Z_1}$

Power dissipated is $\bar{P} = \frac{1}{2} I_0 \mathcal{E}_0 \cos\varphi$

$$\cos\varphi = \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}}$$

so $\bar{P} = \frac{1}{2} I_0^2 (Z_1^2 + Z_2^2)^{1/2} \dfrac{Z_1}{\sqrt{Z_1^2+Z_2^2}}$

$$\bar{P} = \frac{1}{2} I_0^2 Z_1$$

real part of the total impedance