Levi-Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is even permutation of } 123 \\ -1 & \text{if } ijk \text{ is odd permutation of } 123 \\ 0 & \text{otherwise, ie if any two of the } ijk \text{ are equal} \end{cases}$$

ijk is an $\binom{\text{even}}{\text{odd}}$ permutation of 123 if you can get to it from 123 by making an $\binom{\text{even}}{\text{odd}}$ number of pairwise interchanges.

Example: 213 is an odd permutation $\underbrace{12}_{\text{one switch}}3 \rightarrow 213$

231 is an even permutation $\underbrace{12}_{\text{switch}}3 \rightarrow 2\underbrace{13}_{\text{switch}} \rightarrow 231$

If $\vec{A} = \vec{B} \times \vec{C}$ then ith component of $\vec{A}$ is given by

$$A_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} B_j C_k$$

For example, $A_1 = \sum_{j,k} \varepsilon_{1jk} B_j C_k$

all other terms vanish

$$= \varepsilon_{123} B_2 C_3 + \varepsilon_{132} B_3 C_2$$

$$A_1 = B_2 C_3 - B_3 C_2 \quad \text{correct!}$$

Similarly

$$A_2 = \varepsilon_{231} B_3 C_1 + \varepsilon_{213} B_1 C_3$$
$$= B_3 C_1 - B_1 C_3$$

$$A_3 = \varepsilon_{312} B_1 C_2 + \varepsilon_{321} B_2 C_1$$
$$= B_1 C_2 - B_2 C_1$$

A very useful relation is

$$\sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$$

Since $\varepsilon_{ijk} = 0$ unless ijk are all different the above will be non zero only if the pair j,k has the same numbers as the pair l,m.
when $j=l$ and $k=m$, the above is $(\varepsilon_{ijk})^2 = +1$
when $j=m$ and $k=l$, the above is $\varepsilon_{ijk}\,\varepsilon_{ikj} = -1$
You can check that both sides of the above equation obey these properties, hence the equality

__Example:__ $\vec{A} \times (\vec{B} \times \vec{C})$

ith component of above is

$$\sum_{jklm} \varepsilon_{ijk} A_j \underbrace{\varepsilon_{klm} B_l C_m}_{k^{th}\text{ component of } \vec{B}\times\vec{C}}$$

$$= \sum_{jklm} \varepsilon_{kij}\,\varepsilon_{klm} A_j B_l C_m$$

$$= \sum_{jlm} [\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}] A_j B_l C_m$$

$$= \sum_{j} A_j B_i C_j - A_j B_j C_i$$

$$= B_i (\vec{A}\cdot\vec{C}) - C_i (\vec{A}\cdot\vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$