Electrostatics - all source charges are stationary

Coulomb's Law: all of electrostatics is contained in Coulomb's Law - all we do this semester is find different ways to restate Coulomb's law in forms that make it easier to solve problems!

MKS units

force on test charge \( Q \) at \( \vec{r} \) due to source charge \( q \) at \( \vec{r}_0 \)

\[
\vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{r^2} \hat{r}
\]

\[
\vec{r} = \vec{r} - \vec{r}_0
\]

\[
\hat{r} = \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|} \quad \text{unit vector}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{qQ}{|\vec{r} - \vec{r}_0|^3}
\]

charge \( q \) measured in Coulombs

\[\varepsilon_0 = 8.85 \times 10^{-12} \text{ Coul m}^{-2} \text{ N}^{-1} \text{ m}^2 \] permittivity of free space

Electric field: \( \vec{E}(\vec{r}) \) is the force, per unit charge, that would be exerted on a test charge \( Q \) at \( \vec{r} \), due to all the source charges.

for a point source charge \( q \) at \( \vec{r}_0 \),

\[
\vec{E}(\vec{r}) = \frac{\vec{F}}{Q} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{|\vec{r} - \vec{r}_0|^3}
\]

For many charges: Principle of Superposition: force from many charges is just the linear sum of forces from each individual charge.
for charges \( q_i \) at positions \( \mathbf{r}_i \),

\[
\varepsilon_0 \sum_{i=1}^{n} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}
\]

**Continuous charge density**: define **volume charge density**

\[ g(\mathbf{r}) \]

\( g(\mathbf{r}) \Delta V = \text{total charge contained in infinitesimal volume } \Delta V \text{ about position } \mathbf{r} \).

\( g(\mathbf{r}) \) has units charge/vol or coul/m^3.

\[ E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{d^3r'}{|\mathbf{r} - \mathbf{r}'|^3} \]

**Differential volume element**

\[ d^3r \equiv dV = dx \, dy \, dz \]

**Surface charge density** \( \sigma(\mathbf{r}) \): \( \sigma(\mathbf{r}) \Delta A = \text{total charge contained in area } \Delta A \text{ about position } \mathbf{r}, \) on some specified surface \( S \).

\[ E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int da' \sigma(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

\( da \) = differential area element

**Line charge density** \( \lambda(\mathbf{r}) \)

\( \lambda(\mathbf{r}) \Delta L = \text{total charge contained in length } \Delta L \) about \( \mathbf{r} \) on a specified curve \( C \).

\[ E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int dl' \lambda(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

\( dl \) = differential length element
Surface integrals over surface $S'$. The surface with infinitesimally small tiles of area $d\mathbf{a}$ at positions $\mathbf{r}_c$ on $S'$:

$$\int_{S'} d\mathbf{a} \cdot \mathbf{f} (\mathbf{r}) = \sum_{i} \mathbf{f} (\mathbf{r}_c) d\mathbf{a}_i$$

Easy to evaluate analytically only for simple geometries.

Example: $S' = \text{flat rectangular surface}$ at $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, $z = z_0$ constant.

$$\Rightarrow \int_{S'} d\mathbf{a} \cdot \mathbf{f} (\mathbf{r}) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y, z_0) dx dy$$

Vector surface integral:

$$\int_{S'} d\mathbf{a} \cdot \mathbf{U} (\mathbf{r}) = \sum_{i} d\mathbf{a}_i \cdot \mathbf{U}(\mathbf{r}_c)$$

where $\hat{\mathbf{n}}$ is the outward pointing unit vector normal to the surface $S'$ at point $\mathbf{r}$; direction of $\hat{\mathbf{n}}$ will in general vary as position $\mathbf{r}$ on surface varies.

$$\int_{S'} d\mathbf{a} \cdot \mathbf{U} (\mathbf{r}) = \sum_{i} d\mathbf{a}_i \cdot \hat{\mathbf{n}}_i \cdot \mathbf{U}(\mathbf{r}_c)$$

For flat rectangle example above, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ everywhere on $S$:

$$\int_{S'} d\mathbf{a} \cdot \mathbf{U} (\mathbf{r}) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \hat{\mathbf{z}} \cdot U(x, y, z_0) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} U (x, y, z_0)$$
**Line integral along curve C**

- make curve C out of infinitesimally small straight line segments of length \( dl \), located at positions \( \vec{r}_i \) on C.

\[
\int_C dl \cdot \vec{F}(\vec{r}) = \sum_i \vec{F}(\vec{r}_i) \cdot dl_i
\]

**Vector line integral**

- make curve out of infinitesimal displacements \( d\vec{l}_i \); direction of \( d\vec{l}_i \) is tangent to C.

\[
\int_C d\vec{l} \cdot \vec{U}(\vec{r}) = \sum_i \vec{U}(\vec{r}_i) \cdot d\vec{l}_i
\]

It is easy to convert vector line integral into an ordinary one dimensional integral, provided one has a parameterization of the curve C. For example, suppose curve is given by

parameterization \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \)

where \( \vec{r}(t_0) = \vec{r}_0 \) and \( \vec{r}(t_1) = \vec{r}_1 \)

are end points.

Denote the points \( \vec{r}_i = \vec{r}(t_i) \) where \( t_i - t_{i-1} = \Delta t \)

\[
\int_C d\vec{l} \cdot \vec{U}(\vec{r}) = \sum_i \vec{U}(\vec{r}_i) \cdot d\vec{l}_i
\]

By our definition, \( d\vec{l}_i = \vec{r}_i - \vec{r}_{i-1} = \vec{r}(t_i) - \vec{r}(t_{i-1}) \approx \frac{d\vec{r}}{dt} \Delta t \)
\[
\int_{C} d\tau \cdot \vec{u}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} \Delta t = \int_{t_0}^{t_f} dt \frac{d\vec{r}}{dt} \cdot \vec{u}(\vec{r}(t))
\]

Physical example: let \( \vec{r}(t) \) be the trajectory of a particle in time as it moves under the action of a force \( \vec{F} \). The work done on the particle is

work: \( W = \int_{C} d\tau \cdot \vec{F} = \int_{t_0}^{t_f} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt \)

here \( \frac{d\vec{r}}{dt} \) is just the velocity \( \vec{v} \)

\( \Rightarrow W = \int_{t_0}^{t_f} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt \) \hspace{1cm} \text{But} \ \vec{F} \cdot \vec{v} \ \text{is just the power that force expends on moving particle}

\( \Rightarrow W = \int_{t_0}^{t_f} \text{(power)} \, dt \)

So by parameterizing the particle's trajectory in terms of time \( t \), we recover the familiar result from mechanics that

work done = time integral of power