Electrostatics - all source charges are stationary

Coulomb's Law: all of electrostatics is contained in Coulomb's Law - all we do this semester is find different ways to restate Coulomb's law in forms that make it easier to solve problems!

MKS units

force on test charge Q at $\vec{r}$ due to source charge q at $\vec{r}_0$

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}_0$$

$$\hat{\mathscr{r}} = \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|} \quad \text{unit vector from } \vec{r}_0 \text{ to } \vec{r}$$

$$= \frac{1}{4\pi\epsilon_0}\, qQ\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

charge q measured in Coulombs

$\epsilon_0 \equiv 8.85\times10^{-12}\ \frac{\text{coul}^2}{\text{nt}\ m^2}$ permittivity of free space

Electric field: $\vec{E}(\vec{r})$ is the force, per unit charge, that would be exerted on a test charge Q at $\vec{r}$, due to all the source charges.

for a point source charge q at $\vec{r}_0$,

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{Q} = \frac{1}{4\pi\epsilon_0}\frac{q\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\epsilon_0}\, q\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

For many charges: Principle of Superposition: force from many charges is just the linear sum of forces from each individual charge for charges q_i at positions $\vec{r}_i$,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}$$

<u>Continuous charge density</u> : define <u>volume charge density</u> $\rho(\vec{r})$ by

$\rho(\vec{r})\Delta V$ = total charge contained in infinitesmal volume ΔV about position $\vec{r}$.

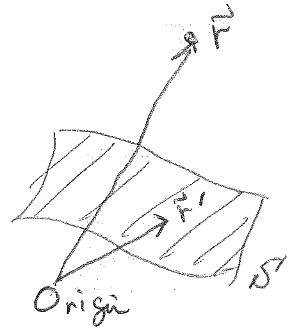
$\rho(\vec{r})$ has units charge/vol or coul/m^3.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$d^3r \equiv dV = dx\,dy\,dz$ differential volume element

<u>surface charge density</u> $\sigma(\vec{r})$: $\sigma(\vec{r})\Delta A$ = total charge contained in area ΔA about position $\vec{r}$, on some specified surface S.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Surface}} da' \, \sigma(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad \vec{r}\,' \in S$$

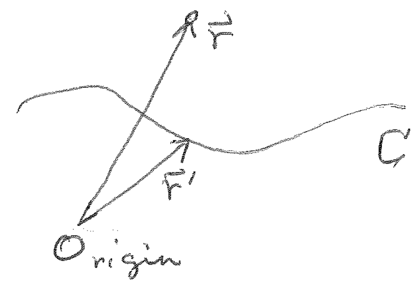


da = differential area element

<u>line charge density</u> $\lambda(\vec{r})$

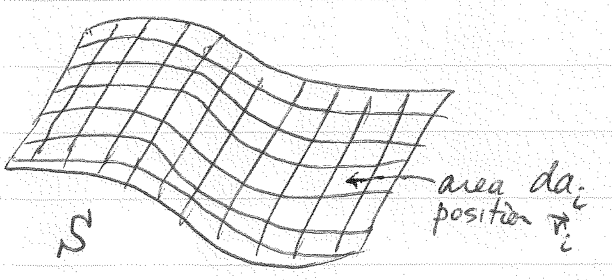
$\lambda(\vec{r})\Delta L$ = total charge contained in length ΔL about $\vec{r}$ on a specified curve C.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Curve}} dl' \, \lambda(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad \vec{r}\,' \in C$$



dl = differential length element

Math Review Surface integrals over surface S



tile surface with infinitesmally small tiles of area da_i at positions $\vec{r}_i$ on S

$$\int_S da\, f(\vec{r}) \equiv \sum_i f(\vec{r}_i)\, da_i$$

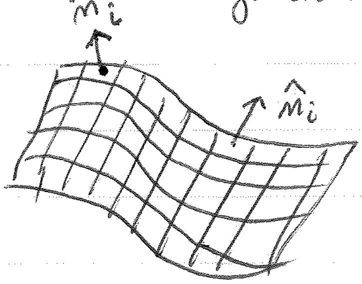
easy to evaluate analytically only for simple geometries.

example S = flat rectangular surface at $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, $z = z_0$ constant.

$$\Rightarrow \int_S da\, f(\vec{r}) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\, f(x, y, z_0)$$

vector surface integral $\displaystyle \int_S d\vec{a} \cdot \vec{u}(\vec{r}) \equiv \int_S da\, \hat{n} \cdot \vec{u}(\vec{r})$

where $\hat{n}$ is the outward pointing unit vector normal to the surface S at point $\vec{r}$; direction of $\hat{n}$ will in general vary as position $\vec{r}$ on surface varies.

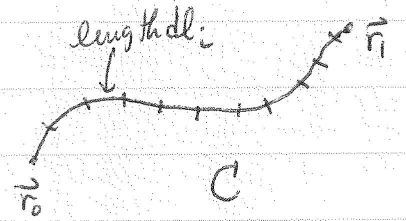


$$\int_S d\vec{a} \cdot \vec{u}(\vec{r}) = \sum_i da_i\, \hat{n}_i \cdot \vec{u}(\vec{r}_i)$$

for flat rectangle example above, $\hat{n} = \hat{z}$ everywhere on S

$$\int_S d\vec{a} \cdot \vec{u}(\vec{r}) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\; \hat{z} \cdot \vec{u}(x, y, z_0) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\; u_z(x, y, z_0)$$

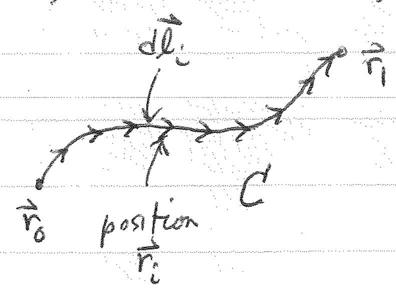
line integral along curve C



make curve C out of infinitesmally small straight line segments of length dl_i located at positions $\vec{r}_i$ on C

$$\int_C dl\, f(\vec{r}) \equiv \sum_i f(\vec{r}_i)\, dl_i$$

vector line integral



make curve out of infintesmal displacements $d\vec{l}_i$; direction of $d\vec{l}_i$ is tangent to C.

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) \equiv \sum_i \vec{u}(\vec{r}_i) \cdot d\vec{l}_i$$

it is easy to convert vector line integral into an ordinary one dimensional integral, provided one has a parameterization of the curve C. For exaple, suppose curve is given by parameterization $\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$

where $\vec{r}(t_0) = \vec{r}_0$ and $\vec{r}(t_1) = \vec{r}_1$ are end points

Denote the points $\vec{r}_i = \vec{r}(t_i)$ where $t_i - t_{i-1} = \Delta t$

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \sum_i \vec{u}(\vec{r}_i)\, d\vec{l}_i$$

By our definition, $d\vec{l}_i = \vec{r}_i - \vec{r}_{i-1} = \vec{r}(t_i) - \vec{r}(t_{i-1}) \simeq \frac{d\vec{r}}{dt}\Delta t$

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \sum_i \vec{u}(\vec{r}(t_i)) \cdot \frac{d\vec{r}(t_i)}{dt} \Delta t$$

$$= \int_{t_0}^{t_1} dt \, \frac{d\vec{r}}{dt} \cdot \vec{u}(\vec{r}(t))$$

Physical example: let $\vec{r}(t)$ be the trajectory of a particle in time as it moves under the action of a force $\vec{F}$. The work done on the particle is

work: $$W = \int_C d\vec{l} \cdot \vec{F} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

here $\frac{d\vec{r}}{dt}$ is just the velocity $\vec{v}$

$$\Rightarrow W = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt$$

But $\vec{F} \cdot \vec{v}$ is just the power that force expends on moving particle

$$\Rightarrow W = \int_{t_0}^{t_1} (\text{power}) \, dt$$

So by parameterizing the particles trajectory in terms of time t, we recover the familiar result from mechanics that

work done = time integral of power