<u>another example</u>: suppose we have a curve C in xy-plane that we can write as $y = y(x)$, $z = z_0$

$\Rightarrow$ can parameterize the curve as

$$\vec{r}(x) = x\,\hat{x} + y(x)\,\hat{y} + z_0\,\hat{z} \qquad \text{from } x = x_0 \text{ to } x = x_1$$

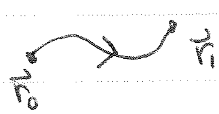
$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \int_{x_0}^{x_1} dx\, \frac{d\vec{r}}{dx} \cdot \vec{u}(\vec{r}(x))$$

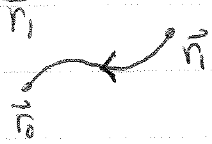
$$= \int_{x_0}^{x_1} dx \left[\hat{x} + \frac{dy}{dx}\,\hat{y} \right] \cdot \vec{u}(x, y(x), z_0)$$

$$= \int_{x_0}^{x_1} dx \left[u_x(x, y(x), z_0) + \frac{dy(x)}{dx}\, u_y(x, y(x), z_0) \right]$$

<u>Note</u>: direction of doing vector line integral is important

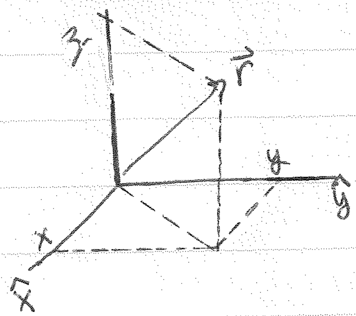
$$\int_{\vec{r}_0}^{\vec{r}_1} d\vec{l} \cdot \vec{u}(\vec{r}) = -\int_{\vec{r}_1}^{\vec{r}_0} d\vec{l} \cdot \vec{u}(\vec{r})$$





Math Review — Coordinate systems - orthonormal, right handed

1) "Cartesian" or "rectangular" coordinates



position vector

$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

any vector

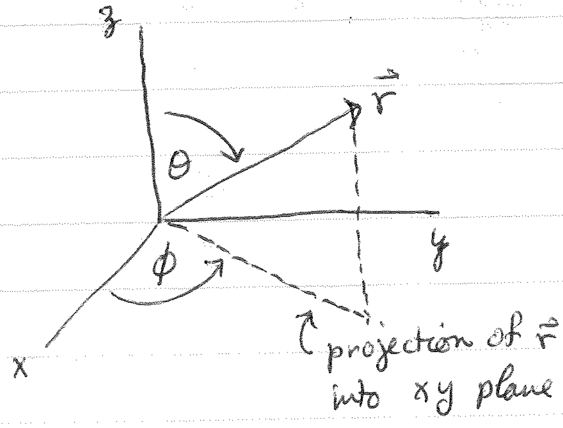
$\vec{A}(\vec{r}) = A_x(\vec{r})\,\hat{x} + A_y(\vec{r})\,\hat{y} + A_z(\vec{r})\,\hat{z}$

differential displacement : $d\vec{r}$

$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

> in some texts, use $\hat{\imath}, \hat{\jmath}, \hat{k}$ instead of $\hat{x}, \hat{y}, \hat{z}$

2) Spherical coordinates (r, θ, ϕ)



if $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, then

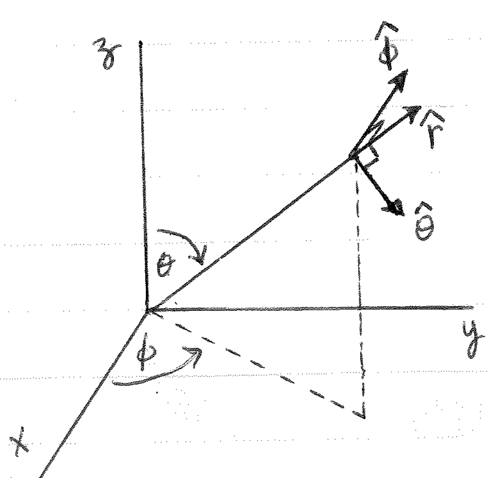
$z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$r = \sqrt{x^2+y^2+z^2} = |\vec{r}|$

unit basis vectors in spherical coords: $\hat{r}, \hat{\theta}, \hat{\phi}$



$\hat{\phi}$ lies in xy plane, ie $\hat{z}\cdot\hat{\phi} = 0$

position vector

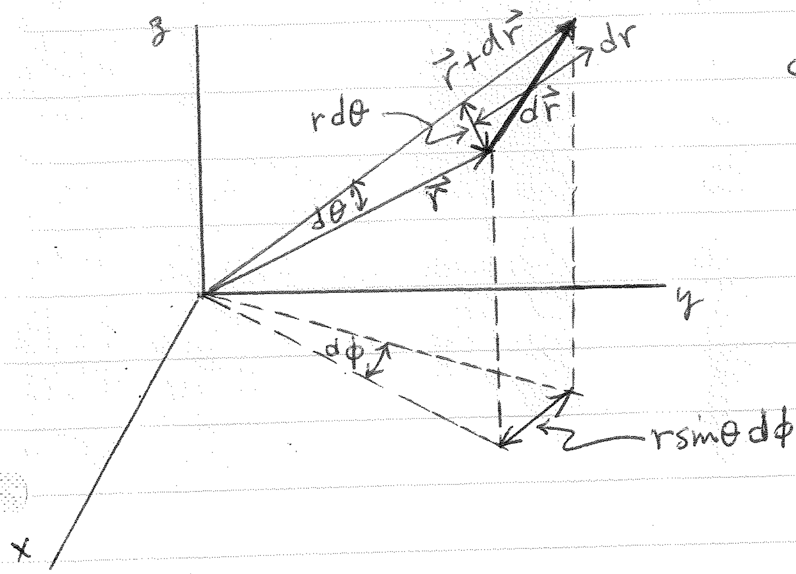
$\vec{r} = r\,\hat{r}$

any vector

$\vec{A}(\vec{r}) = A_r(\vec{r})\,\hat{r} + A_\theta(\vec{r})\,\hat{\theta} + A_\phi(\vec{r})\,\hat{\phi}$

key difference between spherical and Cartesian coords:
the directions of $\hat{x}, \hat{y}, \hat{z}$ is <u>independent</u> of position $\vec{r}$.
the directions of $\hat{r}, \hat{\theta}, \hat{\phi}$ <u>vary</u> as position $\vec{r}$ varies.

<u>differential displacement</u>: $d\vec{r}$



$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

since $\hat{r}, \hat{\theta}, \hat{\phi}$ are orthogonal, the volume swept out as r, θ, ϕ vary by $dr, d\theta, d\phi$ is:

<u>differential volume element</u>:

$$d^3r = (dr)(r\,d\theta)(r\sin\theta\, d\phi) = dr\,d\theta\,d\phi\; r^2\sin\theta$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; f(x,y,z) = \int_0^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\; r^2 \sin\theta\; f(r,\theta,\phi)$$

<u>differential surface element</u> - for surface at fixed radius $r=R$

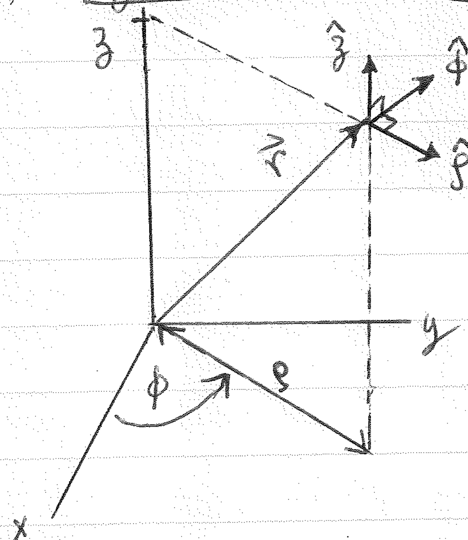
$$da = R^2 \sin\theta\, d\theta\, d\phi$$

$$\int_S da\, f(\vec{r}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\; R^2\sin\theta\; f(R,\theta,\phi)$$

example: surface of sphere of radius R: use above with $f=1$

$$\text{area} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, R^2 \sin\theta = 2\pi R^2 \left(-\cos\theta\right)\Big|_0^{\pi} = 4\pi R^2$$

3) <u>Cylindrical coordinates</u> (ρ, ϕ, z)



$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\Rightarrow \quad x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$\rho = \sqrt{x^2+y^2}$$

unit basis vectors in cylindrical coords: $\hat{\rho}$, $\hat{\phi}$, $\hat{z}$

both $\hat{\rho}$ and $\hat{\phi}$ lie in xy plane. directions of $\hat{\rho}, \hat{\phi}$ depend on position $\vec{r}$

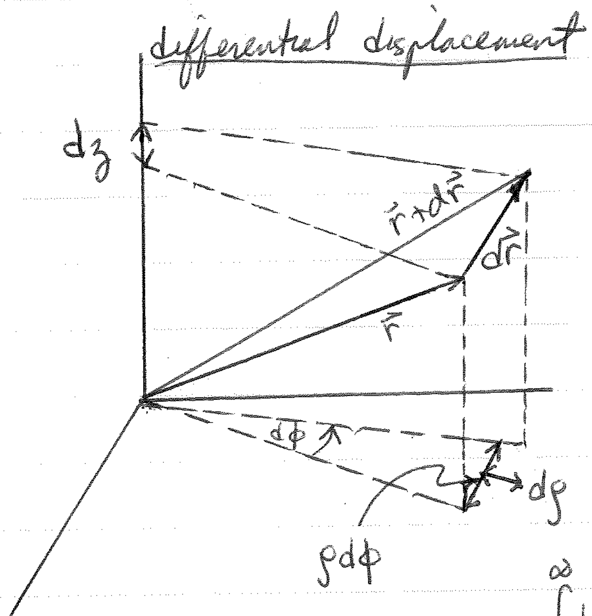
position vector

$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

any vector

$$\vec{A}(\vec{r}) = A_\rho(\vec{r})\,\hat{\rho} + A_\phi(\vec{r})\,\hat{\phi} + A_z(\vec{r})\,\hat{z}$$

<u>differential displacement</u> $d\vec{r}$



$$d\vec{r} = d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

<u>differential volume element</u>:

$$d^3r = (d\rho)(\rho\,d\phi)(dz)$$

$$= d\rho\,d\phi\,dz\,\rho$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; f(x,y,z) = \int_0^{\infty} d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\; \rho\, f(\rho,\phi,z)$$

<u>differential surface element</u> - for surface of cylinder at fixed radius R, and length from $z = L_0$ to $z = L_1$

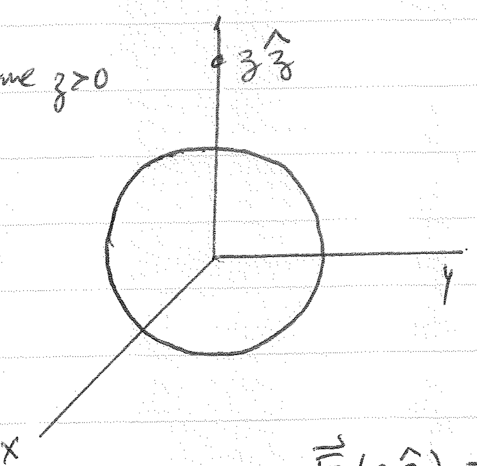
$$da = R \, d\phi \, dz$$

$$\int_S da \, f(\vec{r}) = \int_{L_0}^{L_1} dz \int_0^{2\pi} d\phi \; R \, f(R, \phi, z)$$

Exaple problem 2.7

Find electric field at pt $\vec{r} = z\hat{z}$, from spherical shell of radius R with constant charge density σ_0

assume $z>0$



use $$\vec{E}(\vec{r}) = \int_{S} \frac{da'}{4\pi\epsilon_0}\, \sigma(\vec{r}\,')\, \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

evaluate with $\vec{r} = z\hat{z}$, $\vec{r}\,' = R\hat{r}'$

$\sigma(\vec{r}\,') = \sigma_0$ a constant indep of $\vec{r}\,'$

$da' = d\phi\, d\theta\, R^2 \sin\theta$

$$\vec{E}(z\hat{z}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \frac{R^2 \sin\theta}{4\pi\epsilon_0}\, \sigma_0\, \frac{(z\hat{z} - R\hat{r}')}{|z\hat{z} - R\hat{r}'|^3}$$

when doing the integral, it is <u>crucial</u> to remember that $\hat{r}'$ changes direction as θ and ϕ vary. It is easiest therefore to write $\vec{r}\,' = R\hat{r}'$ in terms of Cartesian coordinates with fixed basis vectors.

$$R\hat{r}' = R\cos\theta\, \hat{z} + R\sin\theta\cos\phi\, \hat{x} + R\sin\theta\sin\phi\, \hat{y}$$

using this we have

$$|z\hat{z} - R\hat{r}'|^3 = \left[(z\hat{z} - R\hat{r}')\cdot(z\hat{z} - R\hat{r}')\right]^{3/2}$$

$$= \left[z^2 + R^2 - 2(z\hat{z})\cdot(R\hat{r}')\right]^{3/2}$$

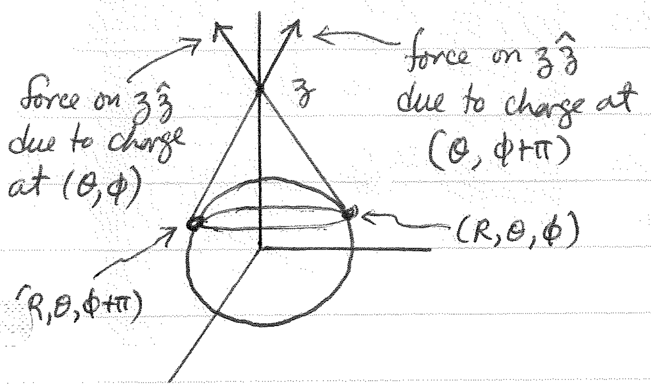
$$= \left[z^2 + R^2 - 2zR\cos\theta\right]^{3/2}$$

$$\vec{E}(z\hat{z}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; \frac{R^2\sigma_0}{4\pi\varepsilon_0} \; \sin\theta \; \frac{\left([z - R\cos\theta]\hat{z} - R\sin\theta[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}] \right)}{[z^2 + R^2 - 2zR\cos\theta]^{3/2}}$$

do integral over ϕ: piece along $\hat{z}$ is indep of $\phi \Rightarrow$ integral gives 2π

$$\left. \begin{array}{l} \text{piece along } \hat{x} \text{ is } \int_0^{2\pi} d\phi \cos\phi = 0 \\ \text{piece along } \hat{y} \text{ is } \int_0^{2\pi} d\phi \sin\phi = 0 \end{array} \right\} \text{ pieces } \perp \hat{z} \text{ vanish.}$$

We could have seen that $\vec{E}(z\hat{z})$ must point along $\hat{z}$ as follows;



consider force on $z\hat{z}$ from elements of charge at points (R,θ,ϕ) and $(R,\theta,\phi+\pi)$. We see from diagram that the components of these forces along $\hat{z}$ are equal, and add; the components $\perp$ $\hat{z}$ are equal but opposite – so they cancel.

$$\vec{E}(z\hat{z}) = 2\pi\hat{z}\,\frac{\sigma_0 R^2}{4\pi\varepsilon_0} \int_0^{\pi} d\theta \; \sin\theta \; \frac{(z - R\cos\theta)}{[z^2 + R^2 - 2zR\cos\theta]^{3/2}}$$

transform variables $\mu = -\cos\theta \Rightarrow d\mu = d\theta\sin\theta$

$$\vec{E}(z\hat{z}) = \frac{\sigma_0 R^2}{2\varepsilon_0}\hat{z} \int_{-1}^{1} d\mu \; \frac{z + R\mu}{[z^2 + R^2 + 2zR\mu]^{3/2}}$$

integrate by parts : $\int d\mu \, u \frac{dv}{d\mu} = uv - \int d\mu \, v \frac{du}{d\mu}$

using $u = (z + R\mu)$ and $v = \frac{-1}{zR(z^2 + R^2 + 2zR\mu)^{1/2}}$

$$\vec{E}(z\hat{z}) = \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left[\left(\frac{-(z+R\mu)}{zR(z^2+R^2+2zR\mu)^{1/2}} \right)_{-1}^{1} + \int_{-1}^{1} d\mu \, \frac{R}{zR(z^2+R^2+2zR\mu)^{1/2}} \right]$$

$$= \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left\{ \left[\frac{-(z+R\mu)}{zR(z^2+R^2+2zR\mu)^{1/2}} \right]_{-1}^{1} + \left[\frac{(z^2+R^2+2zR\mu)^{1/2}}{z^2 R} \right]_{-1}^{1} \right\}$$

for $\mu = 1$, consider $(z^2+R^2+2zR)^{1/2} = \sqrt{(z+R)^2} = z+R$

for $\mu=-1$, consider $(z^2+R^2-2zR)^{1/2} = \sqrt{(z-R)^2} = \begin{cases} z-R & z>R, \; z \text{ outside} \\ R-z & z<R, \; z \text{ inside} \end{cases}$

since $\sqrt{(z-R)^2}$ must be positive

So for $z>R$ we have

$$\vec{E}(z\hat{z}) = \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left\{ \frac{-(z+R)}{zR(z+R)} - \frac{-(z-R)}{zR(z-R)} + \frac{z+R}{z^2R} - \frac{z-R}{z^2R} \right\}$$

$$= \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left\{ -\frac{1}{zR} + \frac{1}{zR} + \frac{2R}{z^2R} \right\} = \boxed{\frac{\sigma_0 R^2}{\varepsilon_0 z^2} \hat{z}}$$

for $z<R$ we have

$$\vec{E}(z\hat{z}) = \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left\{ \frac{-(z+R)}{zR(z+R)} - \frac{-(z-R)}{zR(R-z)} + \frac{z+R}{z^2R} - \frac{R-z}{z^2R} \right\}$$

$$= \frac{\sigma_0 R^2 \hat{z}}{2\varepsilon_0} \left\{ -\frac{1}{zR} - \frac{1}{zR} + \frac{2z}{z^2R} \right\} = 0$$

If we write $Q = 4\pi R^2 \sigma_0$ for the total charge on the spherical shell, we have

$$\vec{E}(z\hat{z}) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 z^2}\,\hat{z} & z > R, \text{ outside shell} \\ 0 & z < R, \text{ inside shell} \end{cases}$$

In general, we can write for <u>any</u> $\vec{r}$ outside the shell,

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}.$$

This is because of the spherical symmetry of the problem – given an arbitrary point $\vec{r}$, we could always choose our coordinate system so that $\vec{r}$ lay on the z-axis.

<u>Notice</u>: (1) for $\vec{r}$ <u>outside</u> the shell, $\vec{E}(\vec{r})$ has exactly the same form as for a point charge Q at the origin. We should expect this result when $z \gg R$, as an observer positioned far from the sphere will not be able to see the details of how the charge is distributed in space – it will look just like a point charge. <u>However</u> it is surprising that $\vec{E}$ has the form of a point charge for <u>any</u> $z > R$. This turns out to be true only because the shell is spherical and σ is uniform.

(2) for $\vec{r}$ inside the sphere, $\vec{E}(\vec{r}) = 0$. We should expect this for the point $\vec{r} = 0$, since the origin is equidistant from all the pieces of charge (ie from all points on the surface of the shell) + so the forces will all cancel out. <u>However</u> it is surprising that $\vec{E} = 0$ for any $z < R$ inside. This turns out to be true only because Coulomb force is $\sim 1/r^2$.

(3) $\vec{E}(z\hat{z})$ is <u>discontinuous</u> as one crosses the charged surface

$\vec{E} = 0$ for z just below R

$\vec{E} = \dfrac{\sigma_0 R^2}{\varepsilon_0 R^2}\hat{z} = \dfrac{\sigma_0}{\varepsilon_0}\hat{n}$ for z just above R (where $\hat{n} = \hat{z}$ is outward normal)

$\Rightarrow$ jump in $\vec{E}$ is $\dfrac{\sigma}{\varepsilon_0}\hat{n}$ as cross the charged surface

this turns out to be true in general for crossing any charged surface, with any (non-uniform) surface charge density $\sigma(\vec{r})$.