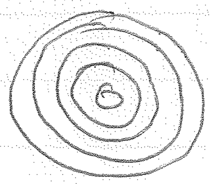
Example: Find electric field from uniformly charged sphere of radius R, + volume charge density ρ.

decompose sphere into concentric shells of width dr.



surface charge on shell at radius r' is

$$\sigma = \rho\, dr'$$

Principle of Superposition

total electric field at pt $\vec{r}$ is sum of electric fields from each shell at radius r', $r' \in [0, R]$

Electric field from shell at radius r' is

$$\vec{E}_{r'}(\vec{r}) = \begin{cases} 0 & |\vec{r}| < r' \\ \dfrac{4\pi r'^2 \rho\, dr'}{4\pi\epsilon_0} \dfrac{\hat{r}}{r^2} & |\vec{r}| > r' \end{cases} \Rightarrow \text{contributes only if } |\vec{r}| \text{ outside shell}$$

total charge on shell is $4\pi r'^2 \rho\, dr'$

total field is $\sum_{r'} \vec{E}_{r'}(\vec{r})$

$$\vec{E} = \int_0^{r'_{max}} dr' \, \frac{4\pi r'^2 \rho}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

where $r'_{max} = R$ if $|\vec{r}| > R$, ie $\vec{r}$ outside sphere
$= |\vec{r}|$ if $|\vec{r}| < R$, ie $\vec{r}$ inside sphere
as shells ~~that are~~ outside $\vec{r}$ contribute no $\vec{E}$ at point $\vec{r}$

$$\vec{E}(\vec{r}) = \frac{\frac{4}{3}\pi r'^{3}_{max}\,\rho}{4\pi\varepsilon_0}\,\frac{\hat{r}}{r^2}$$

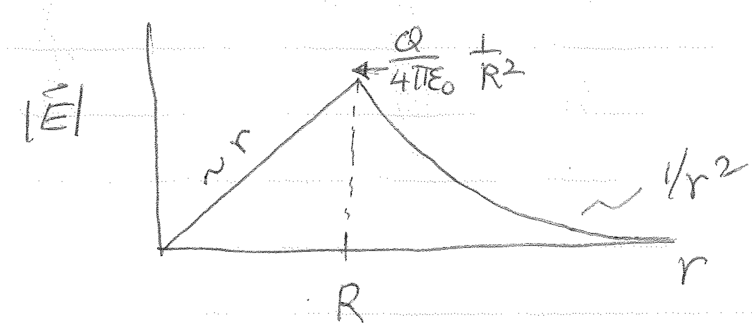
$$= \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0}\,\frac{\hat{r}}{r^2} = \frac{Q}{4\pi\varepsilon_0}\,\frac{\hat{r}}{r^2}$$ if $|\vec{r}| > R$, <u>outside</u>

$Q \equiv \frac{4}{3}\pi R^3$ total charge in sphere

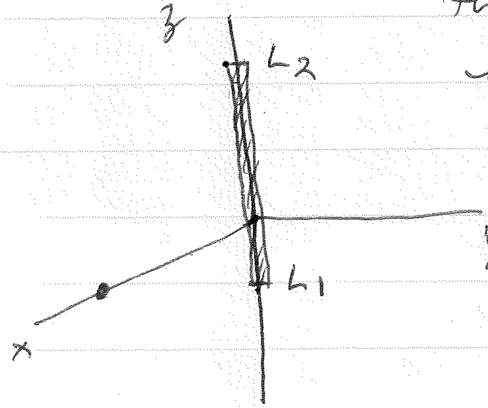
$$= \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\varepsilon_0}\,\frac{\hat{r}}{r^2} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0}\left(\frac{r^3}{R^3}\right)\frac{\hat{r}}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\,\frac{r}{R^3}\,\hat{r}$$ if $|\vec{r}| < R$, <u>inside</u>

Sketch



Example : Find electric field ~~[illegible]~~ at a pt on the x axis, from a thin wire of uniform linear charge density λ_0. The wire lies along the z-axis from $z = L_1$ to $z = L_2$.



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int dl' \, \lambda(\vec{r}\,') \, \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

here $dl' = dz'$, $\vec{r} = x\hat{x}$ since $\vec{r}$ is on x-axis

$\vec{r}\,' = z'\hat{z}$ for $z' \in [L_1, L_2]$

$\lambda(\vec{r}\,') = \lambda_0$ a constant

$$|\vec{r} - \vec{r}\,'|^3 = \left[(x\hat{x} - z'\hat{z})^2 \right]^{3/2} = (x^2 + z'^2)^{3/2}$$

$$\vec{E}(x\hat{x}) = \frac{1}{4\pi\varepsilon_0} \int_{L_1}^{L_2} dz' \, \lambda_0 \left[\frac{x\hat{x} - z'\hat{z}}{(x^2 + z'^2)^{3/2}} \right]$$

the piece along $\hat{x}$ is $\sim \displaystyle\int_{L_1}^{L_2} dz' \frac{1}{(x^2 + z'^2)^{3/2}} = \left[\frac{z'}{x^2 (x^2 + z'^2)^{1/2}} \right]_{L_1}^{L_2}$

(I looked integral up in a handbook!)

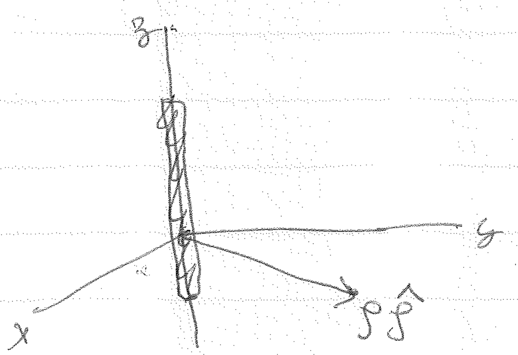
the piece along $\hat{z}$ is $\sim \displaystyle\int_{L_1}^{L_2} dz' \frac{(-z')}{(x^2 + z'^2)^{3/2}} = \left[\frac{1}{(x^2 + z'^2)^{1/2}} \right]_{L_1}^{L_2}$

$$\vec{E}(x\hat{x}) = \frac{\lambda_0}{4\pi\varepsilon_0} \left\{ \left[\frac{L_2}{x(x^2 + L_2^2)^{1/2}} - \frac{L_1}{x(x^2 + L_1^2)^{1/2}} \right] \hat{x} + \left[\frac{1}{(x^2 + L_2^2)^{1/2}} - \frac{1}{(x^2 + L_1^2)^{1/2}} \right] \hat{z} \right.$$

in general, $\vec{E}$ will have component in both $\hat{x}$ and $\hat{z}$ directions.

Because of the rotational symmetry about z axis, we would get same result for any $\vec{r} = \rho\hat{\rho}$ in cylindrical coord.

$$\vec{E} = \frac{\lambda_0}{4\pi\varepsilon_0}\left\{\left[\frac{L_2}{\rho(\rho^2+L_2^2)^{1/2}} - \frac{L_1}{\rho(\rho^2+L_1^2)^{1/2}}\right]\hat{\rho} + \left[\frac{1}{(\rho^2+L_2^2)^{1/2}} - \frac{1}{(\rho^2+L_1^2)^{1/2}}\right]\hat{z}\right\}$$



$\vec{E}$ is indep of ϕ

Check of simple cases

1) Suppose wire is symmetric about origin, $L_1 = -\frac{L}{2}$, $L_2 = \frac{L}{2}$

$$\vec{E}(\rho\hat{\rho}) = \frac{\lambda_0}{4\pi\varepsilon_0}\left\{\frac{L/2}{\rho(\rho^2+\frac{L^2}{4})^{1/2}} - \frac{-L/2}{\rho(\rho^2+\frac{L^2}{4})^{1/4}}\right\}\hat{\rho}$$

piece in $\hat{z}$ vanishes

$$\vec{E} = \frac{\lambda_0}{4\pi\varepsilon_0}\frac{L}{\rho(\rho^2+\frac{L^2}{4})^{1/2}}\hat{\rho}$$

a) for $\rho \ll L$, we can take $\rho^2 + \frac{L^2}{4} \approx \frac{L^2}{4}$

$$\vec{E} \cong \frac{\lambda_0}{4\pi\varepsilon_0}\frac{L}{\rho\left(\frac{L}{2}\right)}\hat{\rho} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{\hat{\rho}}{\rho}$$

result does not depend on L! This is what one would get $\sim \frac{1}{\text{distance from wire}}$ ... [illegible] of infinitely long wire.

b) for $\rho >> L$ we can take $\rho^2 + \frac{L^2}{4} \approx \rho^2$

$$\vec{E} = \frac{\lambda_0}{4\pi\varepsilon_0} \frac{L}{\rho^2} \hat{\rho}$$

just like from point charge
$Q = L\lambda_0$ at origin.
Q is total charge on wire.

What is correction to this result? Do better approximation

$$\left(\rho^2 + \frac{L^2}{4}\right)^{1/2} = \rho\left(1 + \frac{L^2}{4\rho^2}\right)^{1/2} \simeq \rho\left(1 + \frac{L^2}{8\rho^2}\right)$$

using Taylor expansion $\sqrt{1+\epsilon} \simeq 1 + \frac{\epsilon}{2}$

$$\vec{E} \cong \frac{\lambda_0}{4\pi\varepsilon_0} \frac{L\ \hat{\rho}}{\rho^2\left(1+\frac{L^2}{8\rho^2}\right)} \simeq \frac{\lambda_0}{4\pi\varepsilon_0} \frac{L}{\rho^2}\left(1 - \frac{L^2}{8\rho^2}\right)\hat{\rho}$$

using Taylor expansion $\frac{1}{1+\epsilon} \simeq 1 - \epsilon$

$$\vec{E} \cong \frac{\lambda_0}{4\pi\varepsilon_0} \frac{L\ \hat{\rho}}{\rho^2} - \frac{\lambda_0}{4\pi\varepsilon_0} \frac{L^3\ \hat{\rho}}{8\rho^4} + O\left(\frac{1}{\rho^6}\right)$$

first term is like a point charge
we will see that second term is an "electric quadrapole term". second term is smaller than first by a factor $\left(\frac{L}{\rho}\right)^2$.

2) Problem 2.3. Suppose $L_1 = 0$, $L_2 = L$ ie observer is over one end of wire.

$$\vec{E}(\rho\hat{\rho}) = \frac{\lambda_0}{4\pi\varepsilon_0}\left\{ \frac{L}{\rho(\rho^2+L^2)^{1/2}}\hat{\rho} + \left[\frac{1}{(\rho^2+L^2)^{1/2}} - \frac{1}{\rho}\right]\hat{z}\right\}$$

for $\rho >> L$ we take at simplest approx $(\rho^2+L^2)^{1/2} \simeq \rho$

$$\vec{E}(\rho\hat{\rho}) = \frac{\lambda_0}{4\pi\varepsilon_0}\left\{ \frac{L}{\rho^2}\hat{\rho} + \left[\frac{1}{\rho} - \frac{1}{\rho}\right]\hat{z}\right\} = \frac{\lambda_0 L}{4\pi\varepsilon_0}\frac{\hat{\rho}}{\rho^2}$$

like pt charge at origin

better approx: $(\rho^2+L^2)^{1/2} = \rho\left(1+\frac{L^2}{\rho^2}\right)^{1/2} \simeq \rho\left(1+\frac{L^2}{2\rho^2}\right)$

$$\vec{E}(\rho\hat{\rho}) = \frac{\lambda_0}{4\pi\varepsilon_0}\left\{ \frac{L}{\rho^2\left(1+\frac{L^2}{2\rho^2}\right)}\hat{\rho} + \left[\frac{1}{\rho\left(1+\frac{L^2}{2\rho^2}\right)} - \frac{1}{\rho}\right]\hat{z}\right\}$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\left\{ \frac{L}{\rho^2}\left(1-\frac{L^2}{2\rho^2}\right)\hat{\rho} + \left[\frac{1}{\rho}\left(1-\frac{L^2}{2\rho^2}\right) - \frac{1}{\rho}\right]\hat{z}\right\}$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\left[\frac{L}{\rho^2} - \frac{L^3}{2\rho^4}\right]\hat{\rho} - \frac{\lambda_0}{4\pi\varepsilon_0}\frac{L^2}{2\rho^3}\hat{z}$$

$$= \frac{\lambda_0 L}{4\pi\varepsilon_0}\frac{\hat{\rho}}{\rho^2} - \frac{\lambda_0}{4\pi\varepsilon_0}\frac{L^3}{2\rho^4}\hat{\rho} - \frac{\lambda_0}{4\pi\varepsilon_0}\frac{L^2}{2\rho^3}\hat{z}$$

term $\sim \frac{1}{\rho^2}$ is like point charge

term $\sim 1/\rho^3$ is "electric dipole"

term $\sim 1/\rho^4$ is "electric quadrupole"

Maxwell's Equ for electrostatics (Griffiths § 2-2)

We found general solution

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\, \rho(\vec{r}\,')\, \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

For future theoretical development, as well as for sake of finding new + easier ways to solve problems, it is usefull to show that $\vec{E}$ above arises as the solution to a set of partial differential equations that determine $\vec{E}$. These are the Maxwell equ's for electrostatics. They have the form

$$\vec{\nabla}\cdot\vec{E} = \text{something} \qquad \text{divergence of } \vec{E}$$
$$\vec{\nabla}\times\vec{E} = \text{something} \qquad \text{curl of } \vec{E}.$$

Whenever one specifies the divergence + curl of a vector function (as well as specifying the behavior on the boundary of the system) this is enough to uniquely determine the vector function itself. See Helmholtz theorem, Griffiths § 1.6.1

Review of vector differential operators (§ 1-2, 1-3)

Gradient: (1.2.2)

$$f(\vec{r}+d\vec{r}) = f(x+dx,\ y+dy,\ z+dz)$$
$$= f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{df}{dz}dz$$

define gradient vector

$$\vec{\nabla} f(x,y,z) \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$\Rightarrow f(\vec{r}+d\vec{r}) = f(\vec{r}) + (\vec{\nabla} f)\cdot d\vec{r}$ since $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

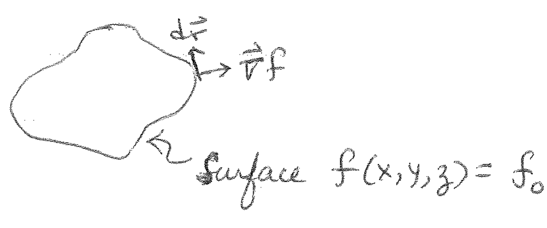
$df \equiv f(\vec{r}+d\vec{r}) - f(\vec{r}) = (\vec{\nabla} f)\cdot(d\vec{r}) = |\vec{\nabla} f|\,|d\vec{r}|\cos\theta$

where θ is angle between $\vec{\nabla} f$ and $d\vec{r}$

$\Rightarrow$ geometrical meanings of gradient:

1) df is max, for a given $|d\vec{r}|$, when $\theta = 0$, ie when $d\vec{r}$ points along $\vec{\nabla} f$. $\Rightarrow$ $\vec{\nabla} f$ points in direction of greatest increase in function f. $|\vec{\nabla} f|$ is the slope of f in this direction

2) $df = 0$ when $\theta = \frac{\pi}{2}$, ie. when $d\vec{r}$ is $\perp$ to $\vec{\nabla} f$. $\Rightarrow$ $\vec{\nabla} f$ is normal to the surfaces of constant f.



3) $df = 0$ for all directions of $d\vec{r}$, if $\vec{\nabla} f = 0$. $\Rightarrow$ $\vec{\nabla} f(\vec{r}_0) = 0$ means $\vec{r}_0$ is a max, min, or saddle pt of $f(\vec{r})$