Differential operator $\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

gradient $f = \vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

<u>Divergence</u> (1.2.4)

apply $\vec{\nabla}$ to a vector function $\vec{v}(\vec{r})$

$$\vec{\nabla}\cdot\vec{v} \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \equiv \text{divergence of } \vec{v}$$

<u>Curl or circulation</u> (1.2.5)

$$\vec{\nabla}\times\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right)$$

$$\cancel{= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)}$$

$$\cancel{= \left(\frac{\partial}{\partial x}v_y + \frac{\partial}{\partial y}v_x\right)\hat{x} + \left(\frac{\partial}{}\right.}$$

$$\equiv \left(\frac{\partial}{\partial y}v_z - \frac{\partial}{\partial z}v_y\right)\hat{x} + \left(\frac{\partial}{\partial z}v_x - \frac{\partial}{\partial x}v_z\right)\hat{y}$$

$$+ \left(\frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x\right)\hat{z}$$

To remember how to compute curl, just think of cyclic permutations of x y z

x y z	$(\vec{A}\times\vec{B})_x$	$= A_y B_z - A_z B_y$
y z x	$(\vec{A}\times\vec{B})_y$	$= A_z B_x - A_x B_z$
z x y	$(\vec{A}\times\vec{B})_z$	$= A_x B_y - A_y B_x$

use rules of differentiation to get product rules like:

$$\vec{\nabla}\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

f: scalar, $\vec{A}$: vector

$$= f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y}$$

$$+ f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}$$

$$\Rightarrow \quad \vec{\nabla}\cdot(f\vec{A}) = f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f$$

or

$$\vec{\nabla}\times(f\vec{A}) = \left[\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right]\hat{x} + \left[\frac{\partial}{\partial z}(fA_x) - \frac{\partial}{\partial x}(fA_z)\right]\hat{y}$$

$$+ \left[\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right]\hat{z}$$

$$= \left[f\frac{\partial A_z}{\partial y} - f\frac{\partial A_y}{\partial z} + \frac{\partial f}{\partial y}A_z - \frac{\partial f}{\partial z}A_y\right]\hat{x}$$

$$+ \left[f\frac{\partial A_x}{\partial z} - f\frac{\partial A_z}{\partial x} + \frac{\partial f}{\partial z}A_x - \frac{\partial f}{\partial x}A_z\right]\hat{y}$$

$$+ \left[f\frac{\partial A_y}{\partial x} - f\frac{\partial A_x}{\partial y} + \frac{\partial f}{\partial x}A_y - \frac{\partial f}{\partial y}A_x\right]\hat{z}$$

$$\Rightarrow \quad \vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}f)\times\vec{A}$$

order of terms is important - it is $(\vec{\nabla}f)\times\vec{A}$ and not $\vec{A}\times(\vec{\nabla}f)$

see sec 1.2.6 for other examples

<u>Second derivatives</u> (1.2.7)

1) <u>Laplacian</u>

$$\vec{\nabla}\cdot(\vec{\nabla} f) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \nabla^2 f$$

2) $$\vec{\nabla}\times(\vec{\nabla} f) = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]\hat{x}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right]\hat{y}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]\hat{z}$$

$$= 0 \quad \text{since } \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \quad \text{etc.}$$

$\vec{\nabla}\times(\vec{\nabla} f) = 0$ for <u>any</u> scalar function $f(\vec{r})$

3) $$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = \hat{x}\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + \hat{y}\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$+ \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$= \hat{x}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z}\right) + \text{etc.}$$

4) $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} = 0.$$

So $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ for any vector function $\vec{v}$

5)
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \hat{x}\left[\frac{\partial}{\partial y}(\vec{\nabla}\times\vec{v})_z - \frac{\partial}{\partial z}(\vec{\nabla}\times\vec{v})_y\right]$$
$$+\hat{y}\left[\frac{\partial}{\partial z}(\vec{\nabla}\times\vec{v})_x - \frac{\partial}{\partial x}(\vec{\nabla}\times\vec{v})_z\right]$$
$$+\hat{z}\left[\frac{\partial}{\partial x}(\vec{\nabla}\times\vec{v})_y - \frac{\partial}{\partial y}(\vec{\nabla}\times\vec{v})_x\right]$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right]$$
$$+\hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$
$$+\hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]$$

$$= \hat{x}\left[-\frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_y}{\partial x\,\partial y} + \frac{\partial^2 v_z}{\partial x\,\partial z}\right]$$
$$+\hat{y}\left[-\frac{\partial^2 v_y}{\partial x^2} - \frac{\partial^2 v_y}{\partial z^2} + \frac{\partial^2 v_x}{\partial y\,\partial x} + \frac{\partial^2 v_z}{\partial y\,\partial z}\right]$$
$$+\hat{z}\left[-\frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_x}{\partial z\,\partial x} + \frac{\partial^2 v_y}{\partial z\,\partial y}\right]$$

$$= \hat{x}\left[-\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial x\,\partial x} + \frac{\partial^2 v_y}{\partial x\,\partial y} + \frac{\partial^2 v_z}{\partial x\,\partial z}\right]$$
$$+\hat{y}\left[-\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial y\,\partial x} + \frac{\partial^2 v_y}{\partial y\,\partial y} + \frac{\partial^2 v_z}{\partial y\,\partial z}\right]$$
$$+\hat{z}\left[-\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial z\,\partial x} + \frac{\partial^2 v_y}{\partial z\,\partial y} + \frac{\partial^2 v_z}{\partial z\,\partial z}\right]$$

$$= \hat{x}\left[-\nabla^2 v_x\right] + \hat{y}\left[-\nabla^2 v_y\right] + \hat{z}\left[-\nabla^2 v_z\right]$$
$$+\ \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \qquad \text{(see (3))}$$

Define Laplacian of vector function

$$\nabla^2 \vec{v} \equiv \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = -\nabla^2 \vec{v} + \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

It is only easy to write what we mean by $\nabla^2 \vec{v}$ in Cartesian coordinates

or we could define more generally

$$\nabla^2 \vec{v} \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{v})$$

could represent these in spherical or cylindrical coords (coming soon!) but result does not look simple

<u>Example</u> $\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ electric field from point charge Q at origin

What is $\vec{\nabla}\cdot\vec{E}$?

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

need to express $\vec{E}$ in Cartesian coordinates to find E_x, E_y, E_z

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} = \frac{Q}{4\pi\varepsilon_0}\frac{(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

Similarly, $\frac{\partial E_y}{\partial y} = \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right)\frac{Q}{4\pi\varepsilon_0}$ and $\frac{\partial E_z}{\partial z} = \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)\frac{Q}{4\pi\varepsilon_0}$

So

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{r^2} - \frac{3r^2}{r^5}\right]$$

$$= 0 \, !$$

So it looks like $\vec{\nabla}\cdot\vec{E} = 0$ for a point charge. This turns out to <u>not</u> be quite correct as we will soon see.

Define Laplacian of vector function

$$\nabla^2 \vec{v} \equiv \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = -\nabla^2\vec{v} + \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

---

gradient, divergence, curl, Laplacian, in spherical + cylindrical coordinates

spherical coords: scaler func $f(r,\theta,\phi)$

gradient $\vec{\nabla} f$

You might think that $\vec{\nabla} f$ should be

$$\frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial \phi}\hat{\phi}$$

using analogy with the form in cartesian coords

$$\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}.$$

This is <u>not</u> correct. To get correct form, use definition:

$$df = f(\vec{r}+d\vec{r}) - f(\vec{r}) \equiv (\vec{\nabla} f)\cdot d\vec{r}$$

in spherical coords:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \qquad (*)$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$df = (\vec{\nabla} f)\cdot d\vec{r}$$

$$\Rightarrow \quad df = (\vec{\nabla} f)_r\,dr + (\vec{\nabla} f)_\theta\,r\,d\theta + (\vec{\nabla} f)_\phi\,r\sin\theta\,d\phi$$

compare with $(*) \Rightarrow (\vec{\nabla} f)_r = \frac{\partial f}{\partial r}, \; (\vec{\nabla} f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}, \; (\vec{\nabla} f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

$$\Rightarrow \quad \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

<u>divergence</u>

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}v_r + \hat{\theta}v_\theta + \hat{\phi}v_\phi\right)$$

you might think this should be:

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

But this is <u>not</u> correct - because when we look at any particular term, for example $\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot(\hat{r}v_r)$ we need to remember that $\hat{r}, \hat{\theta}, \hat{\phi}$ vary with position, + so derivatives of basis vectors must also be taken into account

$$\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot(\hat{r}v_r) = \underbrace{\hat{\theta}\cdot\hat{r}}_{=0}\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_r}{r}\underbrace{\hat{\theta}\cdot\frac{\partial\hat{r}}{\partial\theta}}_{\neq 0}$$

when take this into account correctly, find

$$\boxed{\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}$$

* Note common mistake: $\vec{\nabla}\cdot\vec{v} \neq \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_\phi}{\partial\phi}$!!

Similarly for curl:

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{v} = &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}\\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}\\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}\end{aligned}}$$

Laplacian:

$$\boxed{\begin{aligned}\nabla^2 f = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\end{aligned}}$$

* Note common mistake: $\nabla^2 f \neq \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\phi^2}$!!!

One can similarly find for cylindrical coordinates:

gradient: $\vec{\nabla} f = \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

divergence $\vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

curl: $\vec{\nabla}\times\vec{v} = \left[\frac{1}{\rho}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial\rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial\phi}\right]\hat{z}$

Laplacian $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

Example ① $\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$ electric field from point charge Q at origin

Compute $\vec{\nabla} \cdot \vec{E}$ in spherical coords!

Since $E_\theta = E_\phi = 0$ and $E_r = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}$

we have

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \right)$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r}(1) = 0 \;!$$

same answer as before!

Example ② Consider electric field from sphere of radius R filled with uniform charge density ρ_0

$\Rightarrow$ total charge on sphere is $Q = \frac{4}{3}\pi R^3 \rho_0$

electric field is
$$\vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} & r > R \text{ outside} \\ \frac{Q}{4\pi\epsilon_0} \frac{r\,\hat{r}}{R^3} & r < R \text{ inside} \end{cases}$$

so $\vec{\nabla} \cdot \vec{E} = 0$ for $r > R$ outside, as in Example ①

for $r < R$,

$$\vec{\nabla} \cdot \vec{E} = \frac{Q}{4\pi\epsilon_0 R^3} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \cdot r\right)$$

$$= \frac{Q}{4\pi\epsilon_0 R^3} \frac{1}{r^2} 3r^2 = \frac{Q}{\frac{4}{3}\pi\epsilon_0 R^3}$$

$$\vec{\nabla} \cdot \vec{E} = \rho_0 / \epsilon_0$$

So for uniformly charged sphere,

$$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & r > R \quad \text{outside} \\ \dfrac{\rho_0}{\varepsilon_0} & r < R \quad \text{inside} \end{cases}$$

Now suppose we viewed the point charge of Ξ + ① not as a true point charge, but rather as a sphere of very small radius R with charge density $\rho_0 = \dfrac{Q}{\frac{4}{3}\pi R^3}$

We would then conclude that $\vec{\nabla}\cdot\vec{E}$ for the pt charge was

$$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & r > R \quad \text{outside} \\ \dfrac{Q}{\frac{4}{3}\pi R^3 \varepsilon_0} & r < R \quad \text{inside} \end{cases}$$

as $R \to 0$, approaching the limit of a true point charge, then $\vec{\nabla}\cdot\vec{E} = 0$ ~~almost~~ everywhere except at $r = 0$ where $\vec{\nabla}\cdot\vec{E} = \dfrac{Q}{\frac{4}{3}\pi R^3 \varepsilon_0}$ is growing infinitely large! as $R \to 0$.

The function $\vec{\nabla}\cdot\vec{E}$ above thus has peculiar property that it is zero everywhere except at a single point, at which it is infinite; moreover the integral of $\vec{\nabla}\cdot\vec{E}$ over all space just gives $\dfrac{Q}{\varepsilon_0}$. We will soon see that ~~this~~ a function with this strange behavior is called the Dirac delta function

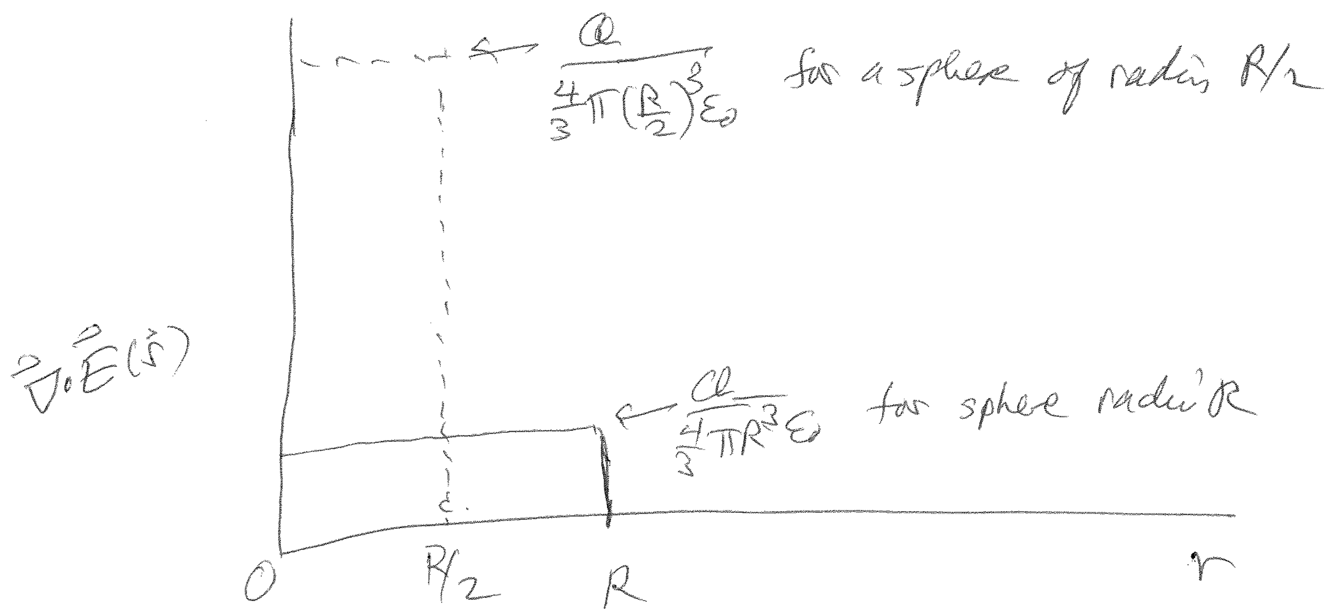
Note: When we directly computed for the point charge

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}(1)$$

this really gives zero only when $r > 0$.

When $r=0$, the above is really an indeterminant expression of the form $\frac{0}{0}$, so we don't know what is its real value.

The preceeding example tells us that the value of $\vec{\nabla}\cdot\vec{E}$ at $r=0$ is really infinite.



as radius of sphere decrease, height of $\vec{\nabla}\cdot\vec{E}$ goes up. As $R \to 0$, $\vec{\nabla}\cdot\vec{E} \to \infty$ at $r=0$