Integral vector calculus (Griffiths §1-3)

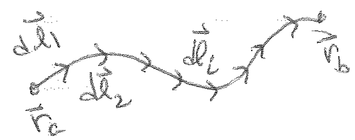
1) Gradients + line Integrals

$$\boxed{\underset{\text{along path } C}{\int_{\vec r_a}^{\vec r_b}} d\vec l \cdot \vec\nabla f = f(\vec r_b) - f(\vec r_a)}$$ independent of path C

proof: for a pt on curve $\vec r$,

$$d\vec l \cdot \vec\nabla f = f(\vec r + d\vec l) - f(\vec r)$$

by definition of gradient



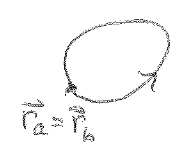
$$\int d\vec l \cdot \vec\nabla f = \left[f(\vec r_a + d\vec l_1) - f(\vec r_a)\right] + \left[f(\vec r_a + d\vec l_1 + d\vec l_2) - f(\vec r_a + d\vec l_1)\right] + \cdots \left[f(\vec r_b) - f(\vec r_b - d\vec l_N)\right]$$

$$= f(\vec r_b) - f(\vec r_a)$$

$$\Rightarrow \oint_C d\vec l \cdot \vec\nabla f = 0$$ if C is a closed path

as then $\vec r_a = \vec r_b$

start = end



2) Divergences + Surface Integrals : Gauss's Theorem

$$\boxed{\int_V d^3x \, [\vec\nabla \cdot \vec v] = \oint_S \vec v \cdot d\vec a}$$

where S is closed surface bounding volume V.

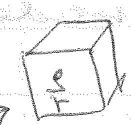
$d\vec a = \hat n \, da$ where $\hat n$ is outward normal to S.

Gauss's theorem provides the geometrical meaning for divergence operator.

$\oint_S \vec{v}\cdot d\vec{a}$ is "flux of v" through the surface S

If $\vec{v}$ represents velocity field of a fluid, then $\oint \vec{v}\cdot d\vec{a}$ gives the total rate that fluid is flowing out the surface S. (see prob 1.32)

divergence $\vec{\nabla}\cdot\vec{v}(\vec{r})$ then gives the flux of $\vec{v}$ out of the pt $\vec{r}$. To see this:



ΔV small, so that $\vec{v}$ is $\approx$ constant over volume ΔV

$$\Rightarrow \int_{\Delta V} d^3r\, \vec{\nabla}\cdot\vec{v} \cong \Delta V\left[\vec{\nabla}\cdot\vec{v}(r)\right] = \oint_S \vec{v}\cdot d\vec{a}$$

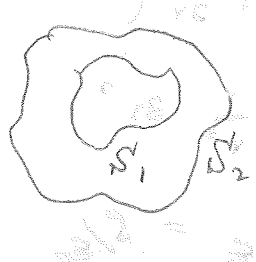
$$\vec{\nabla}\cdot\vec{v}(\vec{r}) \simeq \frac{1}{\Delta V}\oint_S \vec{v}\cdot d\vec{a}$$

$=$ flux per unit volume of $\vec{v}$ out of the point S.

~~3) Curl and line integrals: Stokes Theorem~~

$$\cancel{\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{v}) = \oint_C \vec{v}\cdot d\vec{l}}$$

$\Rightarrow \oint_S \vec{v}\cdot d\vec{a} = 0$ unless $\vec{\nabla}\cdot\vec{v} \neq 0$ somewhere inside S



$$\oint_{S_1} \vec{v}\cdot d\vec{a} = \oint_{S_2} \vec{v}\cdot d\vec{a}$$ provided $\vec{\nabla}\cdot\vec{v} = 0$ in region between S_1 and S_2

Example: Consider $\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ for a point charge at origin. Compute $\oint_S \vec{E}\cdot d\vec{a}$ for the surface S of a sphere of radius R. Since $d\vec{a} = da\,\hat{n} = da\,\hat{r}$ for $\vec{r}$ on S,

$$\vec{E}(\vec{r})\cdot d\vec{a} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{R^2}\cdot da\,\hat{r} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R^2}\,da$$

is constant over surface of sphere S

$$\Rightarrow \oint \vec{E}(\vec{r})\cdot d\vec{a} = \left(\frac{Q}{4\pi\varepsilon_0}\frac{1}{R^2}\right)\underbrace{\left(4\pi R^2\right)}_{\text{area of } S}$$

flux of $\vec{E}$ through S $= \frac{Q}{\varepsilon_0}$ independent of radius R!

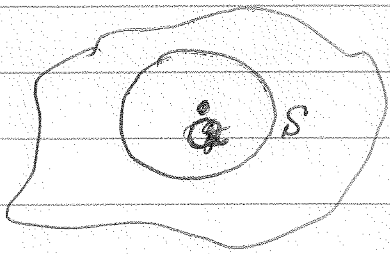
divergence of $\vec{E}$ at origin is therefore

$$\vec{\nabla}\cdot\vec{E}(0) = \frac{1}{\frac{4}{3}\pi R^3}\oint_S E(\vec{r})\cdot d\vec{a} = \frac{Q}{\frac{4}{3}\pi R^3\varepsilon_0} \to \infty$$

as $R \to 0$

This is the same answer we got from our model of the point charge as a uniformly charged sphere of small but finite radius.

Note that it did not matter what was the shape of the surface on which we did the integration



$$\frac{Q}{\varepsilon_0} = \oint_S d\vec{a}\cdot\vec{E}(\vec{r}) = \oint_{S'} d\vec{a}\cdot\vec{E}(\vec{r})$$

where the last equality follows since $\vec{\nabla}\cdot\vec{E} = 0$ everywhere inbetween S and S'

So we have $$\oint_S d\vec{a}\cdot\vec{E} = \begin{cases} \frac{Q}{\varepsilon_0} & \text{for } \underline{\text{any}} \text{ surface } S \text{ that contains the charge } Q \\ 0 & \text{for any surface } S \text{ that does not contain } Q \end{cases}$$

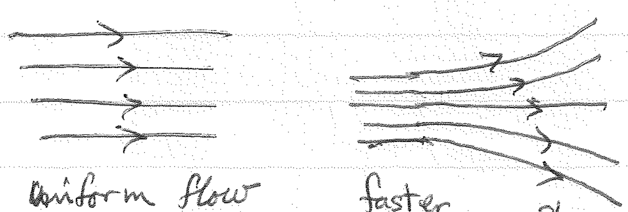
equivalently $$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & \text{for all } \vec{r} \neq 0 \\ \frac{Q}{\Delta V\,\varepsilon_0} \to \infty & \text{at } r=0, \text{ where } \Delta V \text{ is volume of small region containing } Q. \end{cases}$$

$\nabla\cdot\vec{E}$ therefore has the following properties: it vanishes <u>everywhere</u> except at $\vec{r}=0$. At $\vec{r}=0$ it is infinite in just the right way that $\int d^3r\, \vec{\nabla}\cdot\vec{E}$ is a finite constant Q/ε_0. We will see that a function which has this behavior is the "Dirac delta function" "field lines"

draw lines pointed in direction of $\vec{v}(\vec{r})$

density of lines proportional to $|\vec{v}(\vec{r})|$

if $\vec{v}$ is velocity field:



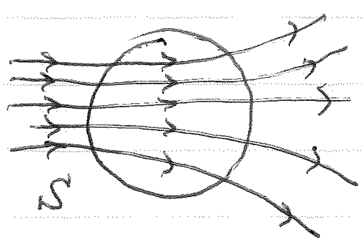
uniform flow   faster flow   slower flow

$\int_S d\vec{a}\cdot\vec{v} \propto$ number of lines passing through S counting with (+) sign if line goes out and with (−) sign if line goes in

If field lines are continuous, then $\vec{\nabla}\cdot\vec{v} = 0$

$\vec{\nabla}\cdot\vec{v} \neq 0$ only at point where field lines become singular.

example

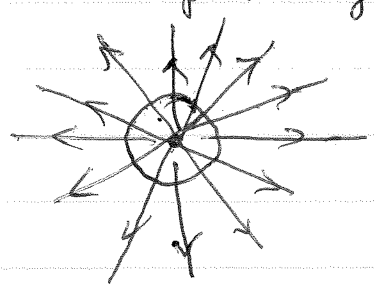


$\oint_S d\vec{a}\cdot\vec{v} = 0$ since field lines continuous, the number of lines going into S = number of lines going out of S. This will be true for any surface S.

$0 = \oint_S d\vec{a}\cdot\vec{v} = \int_V d^3r\ \vec{\nabla}\cdot\vec{v}$ for any S

$\Rightarrow \vec{\nabla}\cdot\vec{v} = 0$

example: $\vec{E}$ from point charge



$\oint_S d\vec{a}\cdot\vec{E} > 0$ as lines only pass out of S.

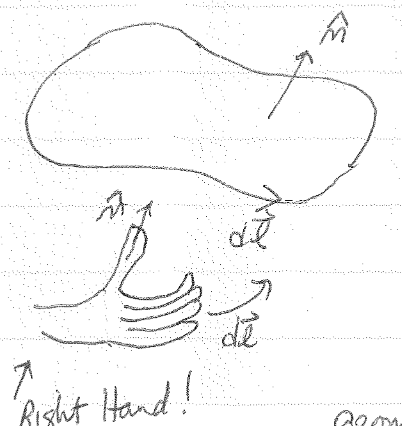
$\Rightarrow \vec{\nabla}\cdot\vec{E} \neq 0$ somewhere inside of S.

(at center where charge is)

3) Curl and Line Integrals : Stokes theorem

$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \oint_C d\vec{l} \cdot \vec{v}$$

where S is an open surface with boundary C



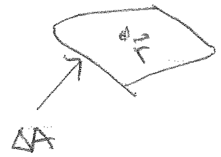
must choose $d\vec{a} = \hat{n}\, da$ and $d\vec{l}$ consistent with right hand rule, i.e. if allign right thumb along $\hat{n}$ (normal to surface) then $d\vec{l}$ must be in direction that fingers point along

geometrical meaning for curl operator

$\oint_C d\vec{l} \cdot \vec{v}$ is the "circulation" of $\vec{v}$ around the loop C

If $\vec{v}$ represents velocity of fluid, and C is a pipe containing fluid (or $\vec{v}$ is vellocity of electrons and C is wire loop) then $\oint_C d\vec{l} \cdot \vec{v}$ gives the net circulation of fluid going around in the loop.

$\vec{\nabla} \times \vec{v}$ gives the circulation of $\vec{v}$ at the pt $\vec{r}$



$$\int_{\Delta A} d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) \cong \Delta A\, \hat{n} \cdot (\vec{\nabla} \times \vec{v}(\vec{r})) = \oint d\vec{l} \cdot \vec{v}$$

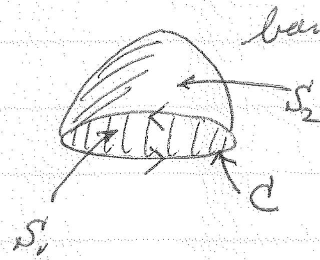
normal to ΔA

$$\hat{n} \cdot \left[\vec{\nabla} \times \vec{v}(\vec{r})\right] = \frac{1}{\Delta A} \oint d\vec{l} \cdot \vec{v}$$ circulation per unit area of $\vec{v}$ at point $\vec{r}$.

$\Rightarrow \int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{v})$ depends only on the boundary line of S

$$\int_{S_1} d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \int_{S_2} d\vec{a} \cdot (\vec{\nabla} \times \vec{v})$$

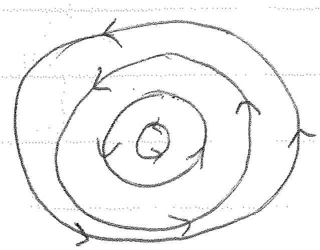
if S_1 and S_2 have same boundary C



$\Rightarrow \oint_S d\vec{a} \cdot \vec{\nabla} \times \vec{v} = 0$ for closed surface S, as boundary curve $C = 0$!

in <u>terms of field lines</u>:

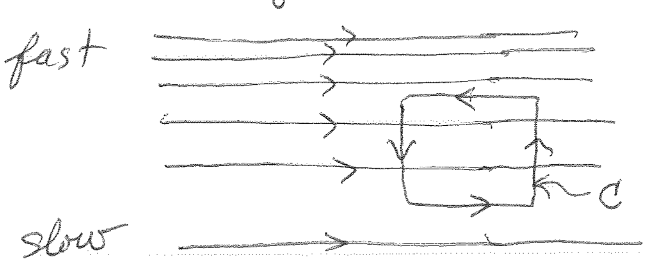
example



Clearly, if we have field-lines that close upon themselves, then we must have $\vec{\nabla} \times \vec{v} \neq 0$ somewhere inside the loop as $\oint d\vec{l} \cdot \vec{v} \neq 0$ along such a curve

But we can also have $\vec{\nabla} \times \vec{v} \neq 0$ in other situations:

Shear flow:



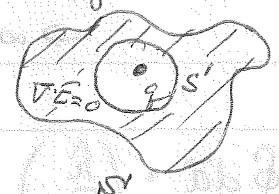
$\oint_C d\vec{l} \cdot \vec{v} \neq 0$!

Gauss Law in Integral form + Differential form

for a pt charge we saw

$$\oint_S d\vec{a}\cdot\vec{E} = \frac{q}{\varepsilon_0}$$

where S can be any surface enclosing charge q – since we know this is true for a spherical surface, and $\vec{\nabla}\cdot\vec{E} = 0$ everywhere except at position of q.



$$\oint_S d\vec{a}\cdot\vec{E} = \oint_{S'} d\vec{a}\cdot\vec{E} = \frac{q}{\varepsilon_0}$$

For many charges, Law of superposition

$$\Rightarrow \boxed{\oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{encl}}{\varepsilon_0}}$$

where Q_{encl} is total amt of charge enclosed by S

Gauss Integral Law

But $Q_{encl} = \int_V d^3r\, \rho(\vec{r})$ where V is vol enclosed by S

$$\oint_S d\vec{a}\cdot\vec{E} = \int_V d^3r\, \frac{\rho(\vec{r})}{\varepsilon_0}$$

$\parallel$ by Gauss's Theorem

$$\int_V d^3r\, \vec{\nabla}\cdot\vec{E} = \int_V d^3r\, \frac{\rho(\vec{r})}{\varepsilon_0}$$

True for any volume V $\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_0}$

Gauss's Differential Law

Dirac δ-function

We saw that for a point charge

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{4\pi\varepsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 0 \quad \text{everywhere except at } \vec{r}=0$$

at $\vec{r}=0$, $\vec{\nabla}\cdot\vec{E}$ is infinite so that

$$\int d^3r\,(\vec{\nabla}\cdot\vec{E}) = \frac{Q}{4\pi\varepsilon_0}\int d^3r\;\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \int d^3r\;\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi$$

This motivates the definition of the Dirac δ-function

$$\delta^3(\vec{r}-\vec{r}\,') = \begin{cases} 0 & \text{everywhere except } \vec{r}=\vec{r}\,' \\ \infty & \text{at } \vec{r}=\vec{r}\,' \end{cases}$$

But $\int_V d^3r'\,\delta^3(\vec{r}-\vec{r}\,') = 1$ for any volume V containing $\vec{r}\,'$,

In ~~terms~~ terms of the δ-function, we can write

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$$

or for a pt charge located at $\vec{r}\,'$

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{\nabla}\cdot\left[\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right] = 4\pi\,\delta^3(\vec{r}-\vec{r}\,')$$