Dirac δ-function (§ 1-5)

we saw $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)=0$ everywhere except at $\vec{r}=0$

But $\oint_S d\vec{a}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi$ for any S that encloses $\vec{r}=0$

$$\parallel$$

$$\int_V d^3r\ \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) \Rightarrow \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\to\infty \text{ at } \vec{r}=0,$$

V enclosed by S $\Rightarrow \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$ is not an ordinary continuous function

This motivates the Dirac δ-function

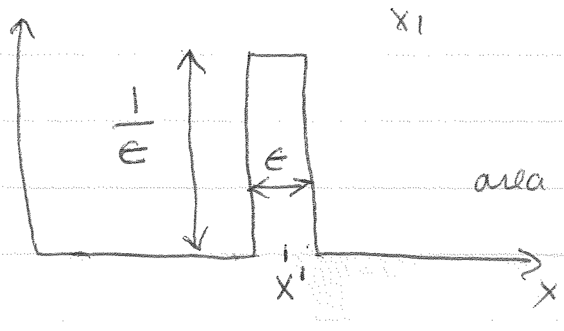
$$\delta(x-x') = \begin{cases} 0 & x\neq x' \\ \infty & x=x' \end{cases}$$

and
$$\int_{x_1}^{x_2} dx\ \delta(x-x') = \begin{cases} 1 & \text{if } x_1<x'<x_2 \\ 0 & \text{otherwise} \end{cases}$$

Can think of $\delta(x-x')$ as being a limit of a sequence of functions:

Let
$$f_\epsilon(x-x') = \begin{cases} 0 & \text{if } |x-x'|>\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & \text{if } |x-x'|<\frac{\epsilon}{2} \end{cases}$$

$$\int_{x_1}^{x_2} dx\ f(x-x') = \begin{cases} 1 & \text{if } x_1<x'-\frac{\epsilon}{2} \text{ and } x_2>x'+\frac{\epsilon}{2} \\ 0 & \text{if } x_2<x'-\frac{\epsilon}{2} \text{ or } x_1>x'+\frac{\epsilon}{2} \end{cases}$$



area under curve = 1

$$\delta(x-x') = \lim_{\epsilon\to 0} f_\epsilon(x-x')$$

Properties of $\delta(x-x')$

$$\int_{x_1}^{x_2} dx \; g(x)\,\delta(x-x') = \begin{cases} g(x') & \text{if } x_1<x'<x_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof: since integrand is zero everywhere except at $x=x'$ we can evaluate $g(x)$ at x' and make it

$$= \int_{x_1}^{x_2} dx \; g(x')\,\delta(x-x') = g(x') \underbrace{\int_{x_1}^{x_2} dx \; \delta(x-x')}_{= \begin{cases} 1 & \text{if } x_1<x'<x_2 \\ 0 & \text{otherwise} \end{cases}}$$

Ex: Consider

$$\int_{-\infty}^{\infty} dx \; g(x)\,\delta(ax+b) = ?$$

let $y = ax+b \qquad dx = \frac{dy}{a}$

if $a>0$

$$= \int_{-\infty}^{\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y) = \frac{g\left(\frac{-b}{a}\right)}{a}$$

if $a<0$

$$= \int_{+\infty}^{-\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y) = -\int_{-\infty}^{\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y)$$

$$= -\frac{g\left(\frac{-b}{a}\right)}{a}$$

general $\Rightarrow$ $$\int_{-\infty}^{\infty} dx \; g(x)\,\delta(ax+b) = \frac{g\left(-\frac{b}{a}\right)}{|a|}$$

$$\Rightarrow \boxed{\delta(ax+b) = \frac{1}{|a|}\,\delta\left(x+\frac{b}{a}\right)}$$

because $\int dx\, g(x)\, \frac{\delta(x+\frac{b}{a})}{|a|} = \frac{1}{|a|}\, g(-\frac{b}{a})$

In general, if $D_1(x)$ and $D_2(x)$ are two expressions involving δ-functions, then $D_1 = D_2$ if

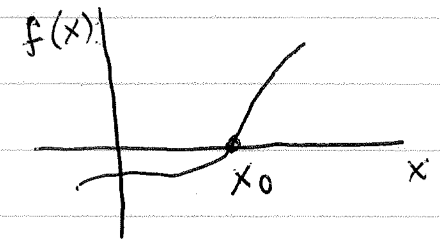
$$\int dx\, g(x)\, D_1(x) = \int dx\, g(x) D_2(x)$$

for any function $g(x)$

Another property of the Dirac δ-function

$$\int_{x_1}^{x_2} dx\, g(x)\, \delta(f(x)) = \frac{g(x_0)}{\left|\frac{df(x_0)}{dx}\right|}$$

if f is monotonic increasing or decreasing with x_0 such that $f(x_0)=0$ and $x_1 < x_0 < x_2$



To see this, note that the only place the integrand is non-zero is when the argument of the δ-function vanishes, ie when $f(x)=0$. This happens at $x=x_0$. So we can expand $f(x)$ in Taylor series about x_0. To lowest order we have

$$f(x) \simeq f(x_0) + \left(\frac{df(x_0)}{dx}\right)(x-x_0) = \frac{df(x_0)}{dx}\,(x-x_0)$$

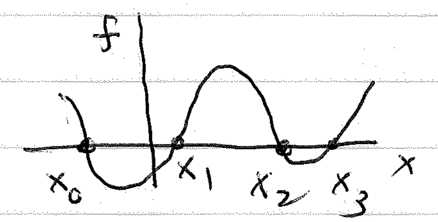
since $f(x_0)=0$. $f(x)$ now has the form

$f(x) = ax + b$ with $a = \frac{df(x_0)}{dx}$ and $b = -\left(\frac{df(x_0)}{dx}\right)x_0$

So from previous example we get

$$\int dx\, g(x)\, \delta(f(x)) = \frac{g(x_0)}{\left|\frac{df(x_0)}{dx}\right|}$$

For a more general $f(x)$ that is not monotonic and may have several zeros at $x_0, x_1, x_2, \ldots$ we have



$$\int_{x_a}^{x_b} dx\, g(x)\, \delta(f(x)) = \sum_{\substack{i \\ \text{such that} \\ x_a < x_i < x_b}} \frac{g(x_i)}{\left|\frac{df(x_i)}{dx}\right|}$$

3-dimensional δ-function

$$\delta^3(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int_V f(\vec{r})\, \delta^3(\vec{r}-\vec{r}\,')\, d^3r = \begin{cases} f(\vec{r}\,') & \text{if } \vec{r}\,' \in V \\ 0 & \text{if } \vec{r}\,' \notin V \end{cases}$$

Recall we said

$$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{\imath}}{\imath^2}\right) = \nabla\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\, \delta^3(\vec{r}-\vec{r}\,')}$$

very important result to remember!!

since $\vec{\nabla}\cdot\left(\frac{\hat{\imath}}{\imath^2}\right) = 0$ except at $\vec{\imath} = \vec{r}-\vec{r}\,' = 0$

and $$\int_V d^3r\, \vec{\nabla}\cdot\left(\frac{\hat{\imath}}{\imath^2}\right) = \oint_S d\vec{a}\cdot\left(\frac{\hat{\imath}}{\imath^2}\right) = \begin{cases} 4\pi & \text{if } V \text{ contains } \vec{r}\,' \\ 0 & \text{otherwise} \end{cases}$$

Now we saw in workshop

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$$

In workshop we did this calculation in Cartesian coords.

Now to see this we can do the differentiation in spherical coordinates

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{\hat{\theta}}{r}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)}_{=0} + \frac{\hat{\varphi}}{r\sin\theta}\underbrace{\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\right)}_{=0}$$

$$= \frac{-\hat{r}}{r^2}$$

So

$$\vec{\nabla}\cdot\left(\vec{\nabla}\left(\frac{1}{r}\right)\right) = \vec{\nabla}\cdot\left(\frac{-\hat{r}}{r^2}\right) = -4\pi\delta^3(\vec{r})$$

or

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})}$$

very important to remember!

Examples of δ-functions in electrostatics

What is volume charge density ρ from set of pt charges q_i at positions $\vec{r}_i$?

Want $\int_V d^3r \ \rho(\vec{r}) =$ total Q enclosed by V

$$= \sum_{\substack{i \\ \text{such that} \\ \vec{r}_i \in V}} q_i$$

$\Rightarrow \boxed{\rho(\vec{r}) = \sum_i q_i \ \delta(\vec{r} - \vec{r}_i)}$ – can check that it has the desired ~~property~~ property above

What is ρ for a ~~...~~ surface charge density laying in xy plane at $z = z_0$?

$$\rho(\vec{r}) = \sigma(x,y) \ \delta(z - z_0)$$

what is ρ for a line charge density along x axis at $y = y_0$, $z = z_0$

$$\rho(\vec{r}) = \lambda(x) \ \delta(y - y_0) \ \delta(z - z_0)$$

what are the units of $\delta(y - y_0)$? They are $\frac{1}{\text{length}}$ so that $\int dy \ \delta(y - y_0) = 1$ is dimensionless

$$\delta^3(\vec{r} - \vec{r}_0) \text{ has units } \frac{1}{(\text{length})^3} = \frac{1}{\text{vol}}$$

Back to Electric field

Using this notation:

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\cdot \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\,\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\,4\pi\delta^3(\vec{r}-\vec{r}')$$

$$= \frac{1}{\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')$$

$$= \frac{1}{\varepsilon_0}\rho(\vec{r})$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$ as we found before

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Curl of $\vec{E}$

$$\vec{\nabla}\times\vec{E}(\vec{r}) = \vec{\nabla}\times \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\,\vec{\nabla}\times\left(\frac{\vec{r}-\vec{r}'}{|r-r'|^3}\right)$$

evaluate in spherical coordinates centered at $\vec{r}'$

$\vec{\nabla}\times\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = \vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0$ since $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ is a vector function with only a radial component, that depends only on radial direction.

ie $\vec{\nabla}\times\vec{v}$ involves derivatives $\frac{\partial v_r}{\partial\theta}, \frac{\partial v_r}{\partial\phi}, \frac{\partial v_\theta}{\partial r}, \frac{\partial v_\theta}{\partial\phi}, \frac{\partial v_\phi}{\partial r}, \frac{\partial v_\phi}{\partial\theta}$ all of which are zero ~~in our~~ when $\vec{v} = \frac{\hat{r}}{r^2}$

$\Rightarrow \vec{\nabla}\times\vec{E} = 0$

Maxwell's Eqn for electrostatics

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} = 0}$$

Gauss' Law

Helmholtz Theorem § 1-6

If the curl + divergence of a vector field are given, one can always solve for the vector field.

if $\vec{\nabla}\cdot\vec{F} = D(\vec{r})$ $\quad \vec{\nabla}\times\vec{F} = \vec{C}(\vec{r})$ $\quad (\Rightarrow \vec{\nabla}\cdot\vec{C} = 0)$

given scalar func   given vector function

then the solution is given by $\vec{F} = -\vec{\nabla}U + \vec{\nabla}\times\vec{W}$

where

$$U(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

provided $D(\vec{r}\,')$, $\vec{C}(\vec{r}\,')$ $\to 0$ as $\vec{r}\,' \to \infty$

Proof:

$=0$ since div of curl $=0$ for any vector function

$$\vec{\nabla}\cdot\vec{F} = -\vec{\nabla}\cdot\vec{\nabla}u + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W})$$
$$= -\nabla^2 u$$
$$= \frac{-1}{4\pi}\int d^3r'\, D(\vec{r}')\, \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$
$$= \frac{1}{4\pi}\int d^3r'\, D(\vec{r}')\, 4\pi\,\delta^3(\vec{r}-\vec{r}') = D(\vec{r})$$

so $\vec{F}$ has the desired divergence

$$\vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\vec{\nabla}u + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) \qquad (\vec{\nabla}\times\vec{\nabla}u = 0 \text{ for any scalar } u)$$
$$= \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = -\nabla^2 W + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$

Consider the 1st term

$$-\nabla^2\vec{W} = -\frac{1}{4\pi}\int d^3r'\, \vec{C}(\vec{r}')\, \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$
$$= \frac{1}{4\pi}\int d^3r'\, \vec{C}(\vec{r}')\, 4\pi\,\delta^3(\vec{r}-\vec{r}') = \vec{C}(\vec{r})$$

Now consider the 2nd term. We hope to find $\vec{\nabla}(\vec{\nabla}\cdot\vec{W}) = 0$

$$\vec{\nabla}\cdot\vec{W} = -\frac{1}{4\pi}\int d^3r'\, \vec{\nabla}\cdot\left[\vec{C}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right]$$

we apply our divergence of a product rule

$$\vec{\nabla}\cdot(f\vec{A}) = (\vec{\nabla}f)\cdot\vec{A} + f(\vec{\nabla}\cdot\vec{A})$$

Here the vector function $\vec{C}(\vec{r}')$ is independent of $\vec{r}$ so $\vec{\nabla}\cdot\vec{C}(\vec{r}') = 0$ and

$$\vec{\nabla}\cdot\left[C(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right] = \vec{C}(\vec{r}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

Next we note by symmetry that

$$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

differentiates with respect to $\vec{r}$ differentiates with respect to $\vec{r}'$

So

$$\vec{\nabla}\cdot\vec{W} = \frac{1}{4\pi}\int d^3r'\, \vec{C}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

Now we want to do a vector integration by parts.
Again use product differentiation rule

$$\vec{\nabla}'\cdot\left(\vec{C}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right) = \left(\vec{\nabla}'\cdot\vec{C}(\vec{r}')\right)\frac{1}{|\vec{r}-\vec{r}'|} + \vec{C}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

So $$\vec{\nabla}\cdot\vec{W} = \frac{1}{4\pi}\int d^3r'\,\vec{\nabla}'\cdot\left(\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{4\pi}\int d^3r'\frac{\left(\vec{\nabla}'\cdot\vec{C}(\vec{r}')\right)}{|\vec{r}-\vec{r}'|}$$

Now the 2nd term above vanishes as we said that since $\vec{\nabla}\times\vec{F} = \vec{C}$, and $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ always, then $\vec{\nabla}\cdot\vec{C} = 0$

For the 1st term we can use Gauss' Theorem to convert it to a surface integral

$$\vec{\nabla}\cdot\vec{W} = \frac{1}{4\pi}\oint_S d\vec{a}'\cdot\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

But if we let our volume of integration be all of space, the S is the bounding surface at infinity. We assumed that sources $D(\vec{r})$ and $\vec{C}(\vec{r})$ were <u>localized</u> ie they vanish as $\vec{r}\to\infty$, so the surface integral above vanishes and $\vec{\nabla}\cdot\vec{W} = 0$.

So $\vec{\nabla}\times\vec{W} = \vec{C}(\vec{r})$ as desired

(can also show that solution we found is unique if require $\vec{F}\to 0$ as $\vec{r}\to\infty$)